# Why conduct seakeeping analysis?

Determine the motions of a design in conditions it is likely to encounter

- Is the vessel going to survive?
- Can the vessel carry out specified task or mission?
- Decide if motions are acceptable:

  Slamming, Deck Wetness, Speed Loss, Human Performance, Ride Control

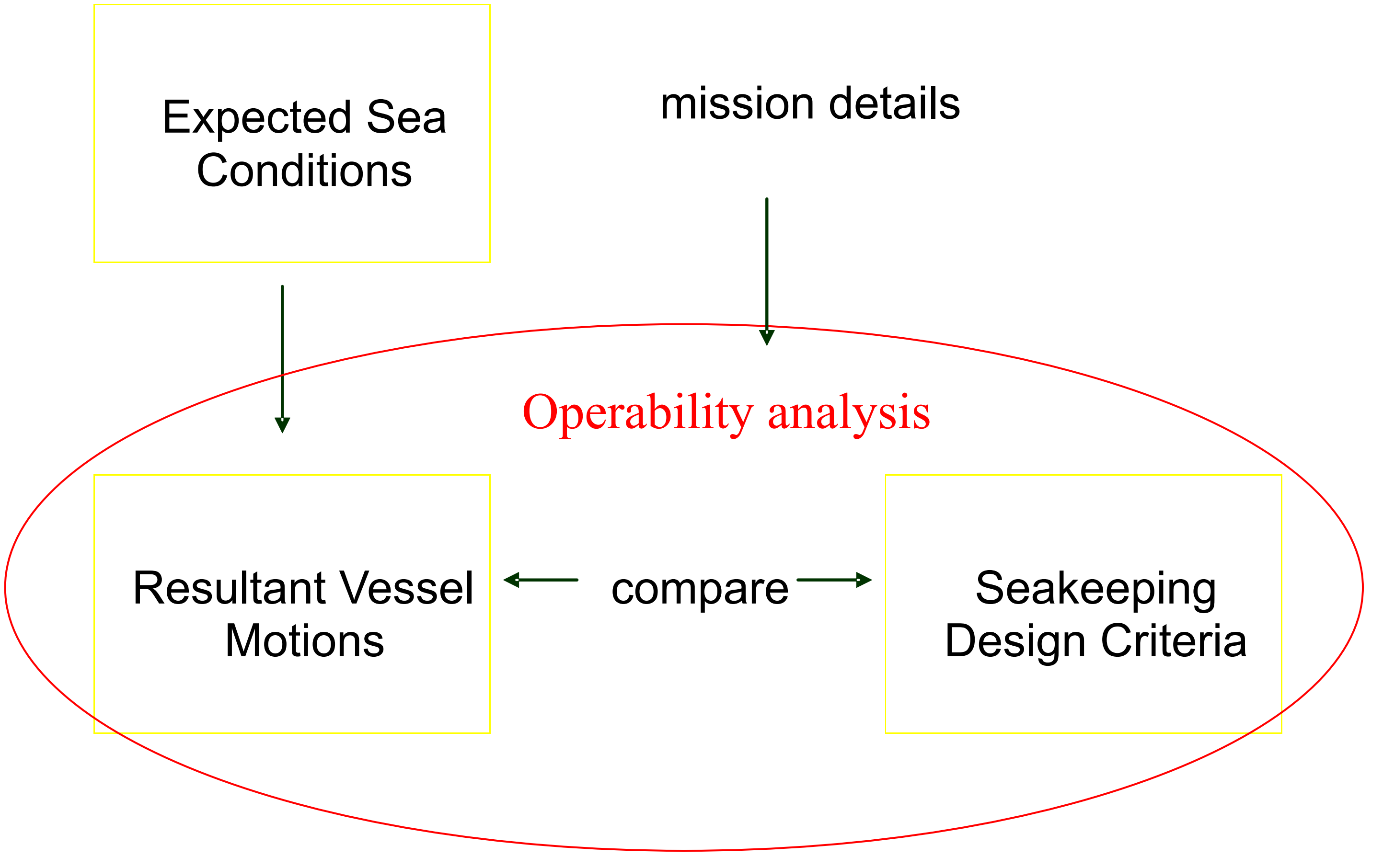
- Decide which design is going to perform the best:

  Design selection, marketing

- What do we already know?
  - The motion of a ship at each given wave frequency (the RAO)
  - How to describe individual waves mathematically

- What do we need to know next?
  - The "average" motion of a ship in a mixture of waves

# Seakeeping Analysis



# Real ocean waves
# content

1. Irregular waves

   Superposition, significant wave height

2. Wave spectra

   Spectral statistics, stylised spectra, encounter spectra, broadness, directionality

3. Probability distributions and extreme waves

   Rayleigh and Gaussian distributions, extreme wave height probability, joint probabilities

# A typical seakeeping specification:

*“ The vessel must not roll more than 10 degrees in sea state 5”*

What is wrong with this?

| Sea state no. | $H_{sig}$ (m) range |
|---|---|
| 0-1 | 0–0.1 |
| 2 | 0.1-0.5 |
| 3 | 0.5-1.25 |
| 4 | 1.25-2.5 |
| 5 | 2.5-4 |
| 6 | 4-6 |
| 7 | 6-9 |
| 8 | 9-14 |

# © "Sea state"

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| 7 | 6-9 |
| 8 | 9-14 |

Ref: World Meteorological Organisation

Ambiguous:

- Large range
- $H_{sig}$ or $H_{mo}$?
- What about period?

# A note on symbols

- Standard ITTC symbols are used where possible

$\omega$ = wave frequency (rad/s)

f = wave frequency (Hz)

- σ is not used for frequency, to avoid confusion with standard deviation
- $C_w$ = wave phase velocity = wave celerity

# Real ocean waves content

1. Irregular waves

   Superposition, significant wave height

2. Wave spectra

   Spectral statistics, stylised spectra, encounter spectra, broadness, directionality

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   Rayleigh and Gaussian distributions, extreme wave height probability, joint probabilities

# 2. Irregular waves
# Objectives

By the end of this section you should be able to:

- Develop an irregular wave time series from regular waves.
- Distinguish between the different methods of describing irregular wave characteristics.

But first: why are we doing all this?...........

# Some temporary assumptions. For the purposes of this segment

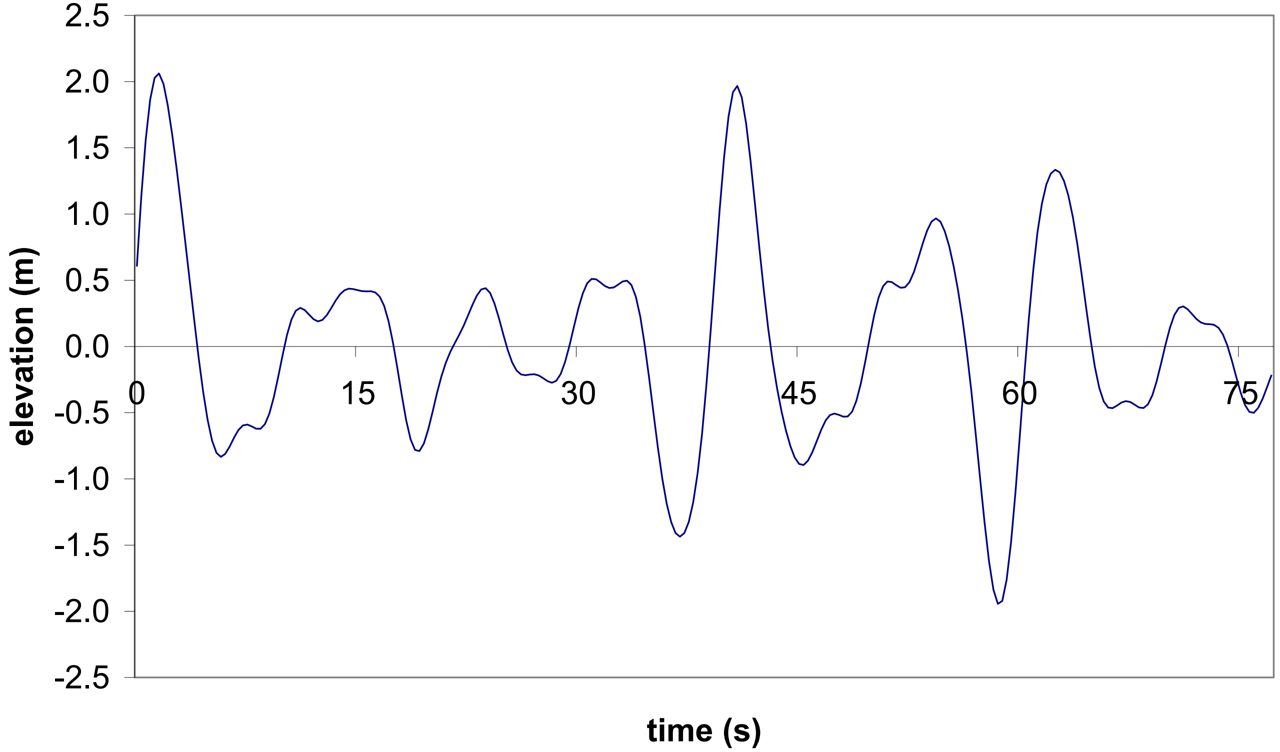
- Waves are all deep water waves.
- Waves are all coming from the same direction.
- Waves are not breaking i.e. low wave steepness
- Linear wave theory is used

# How do we calculate the r.m.s. pitch of a given ship in a given sea state?

We need:

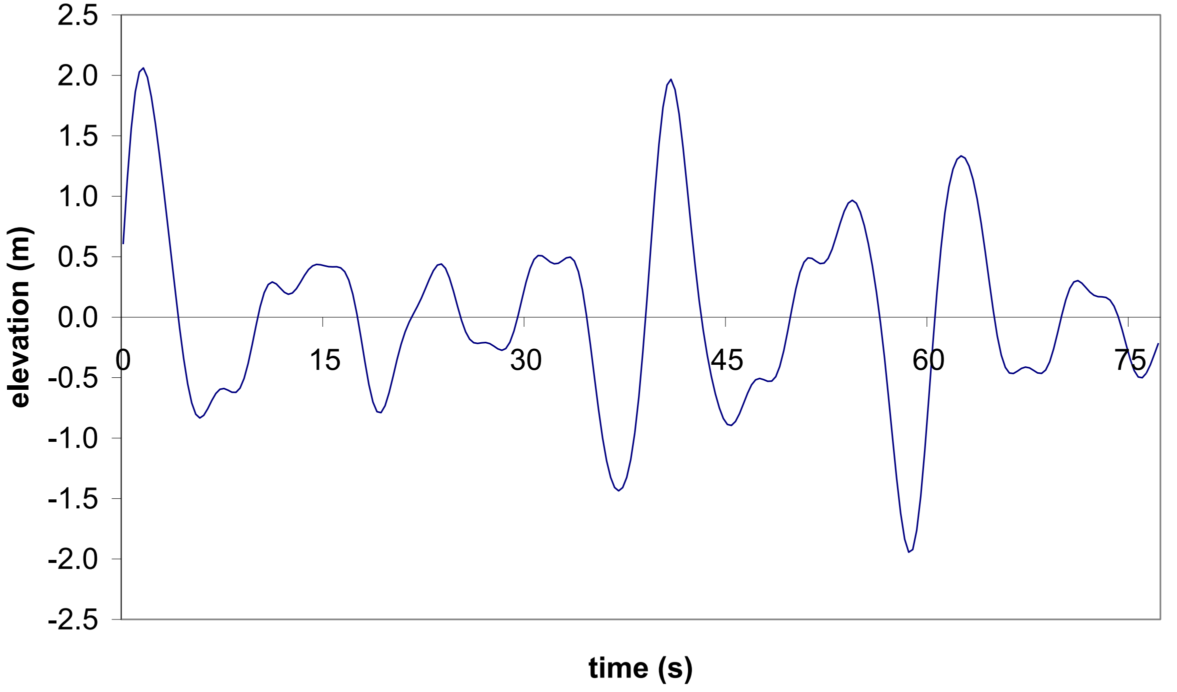
- Pitch RAO (transfer function)
- Mathematical description of the waves
- A method for putting the two together

# Irregular waves




- The simple definitions of height and period used for regular waves become less clear for a realistic irregular wave field.

# Irregular waves – amplitude and height




- Wave amplitude = vertical distance from mean water level to a peak or trough
- Wave height = vertical distance from a trough to a successive peak
- However, average height is not twice the average amplitude; why not?

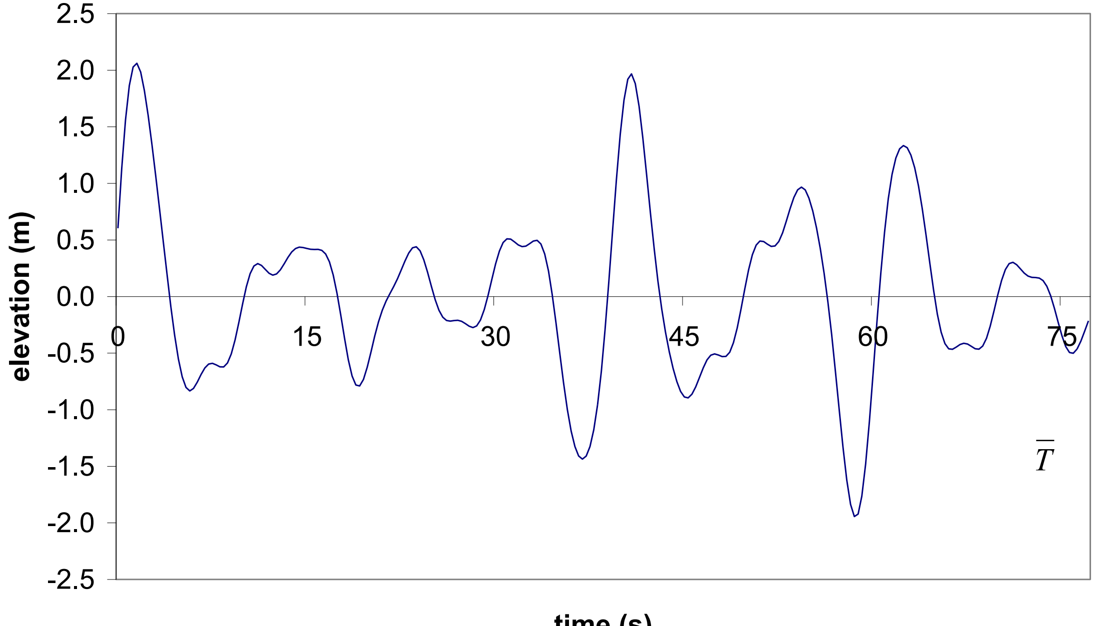
# Irregular waves – significant wave height

The most common definition of wave height is the *significant wave height* $H_{1/3}$ or $H_{sig}$. This is the average of the one-third highest waves.

Example: if there are 99 wave heights in a wave record, list them in height order, take the highest 33 waves (discard the rest) and average these 33 waves. The answer is the significant wave height.

Why average the one-third highest? Why not average all of them?

# Irregular waves – period




- Periods of irregular waves have even more options for defining them The three most common ones are.
- $T_{01}$ or $\overline{T}$ = mean period
- $T_{02}$ or $T_z$ = average period between successive upward crossings of the zero datum (mean level)
- $T_M$ or $T_p$ = the modal period (period with greatest energy)
- $T_{02}$ ($T_z$) is the most commonly used (why?).

# How can we use sine waves to describe the real ocean?

Consider three wave trains with slightly different wave numbers (lengths) , wave frequencies and phase

$$\zeta_1 = \frac{H_1}{2}\cos(k_1 x - \omega_1 t + \varepsilon_1)$$

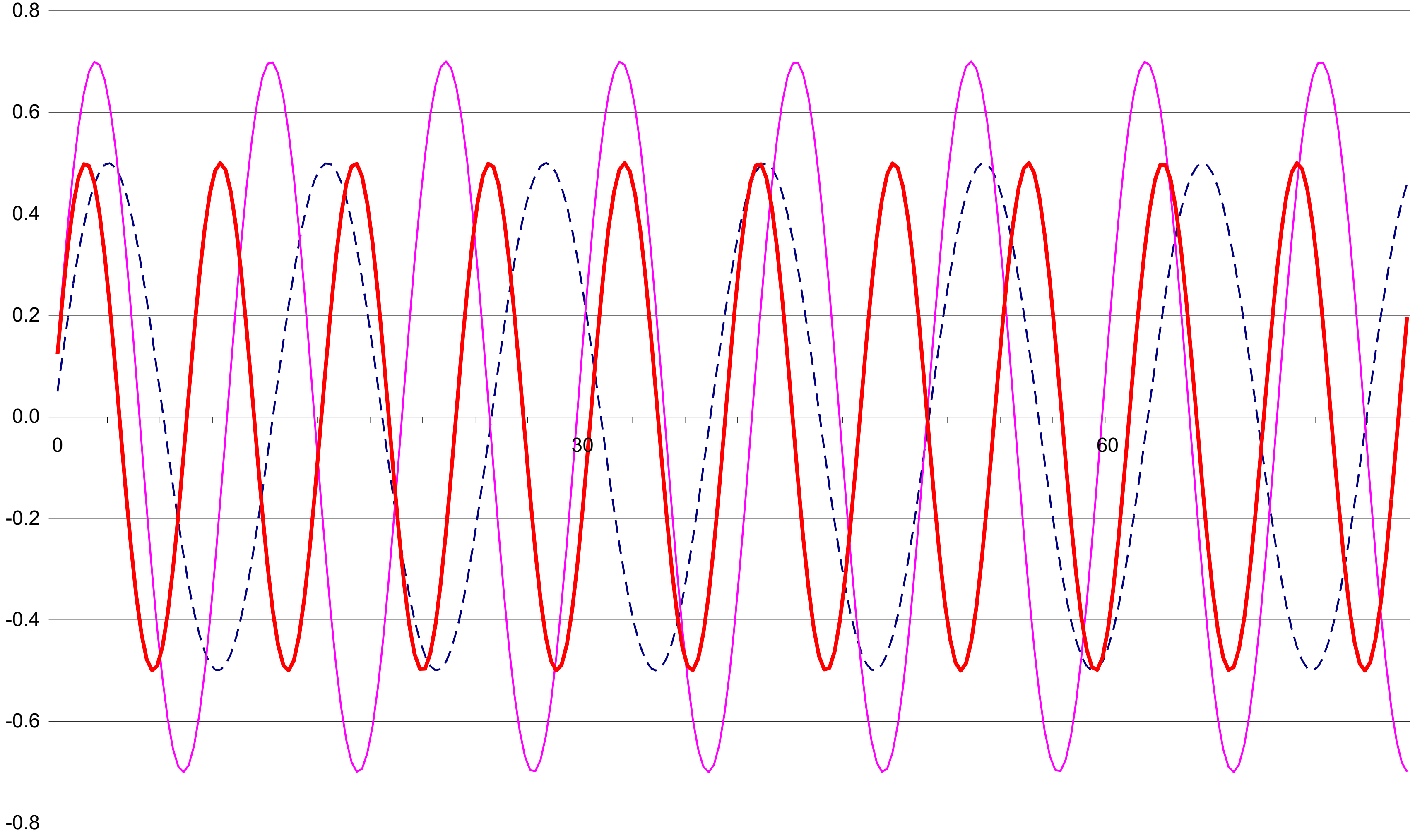
$$\zeta_2 = \frac{H_2}{2}\cos(k_2 x - \omega_2 t + \varepsilon_2)$$

$$\zeta_3 = \frac{H_3}{2}\cos(k_3 x - \omega_3 t + \varepsilon_3)$$

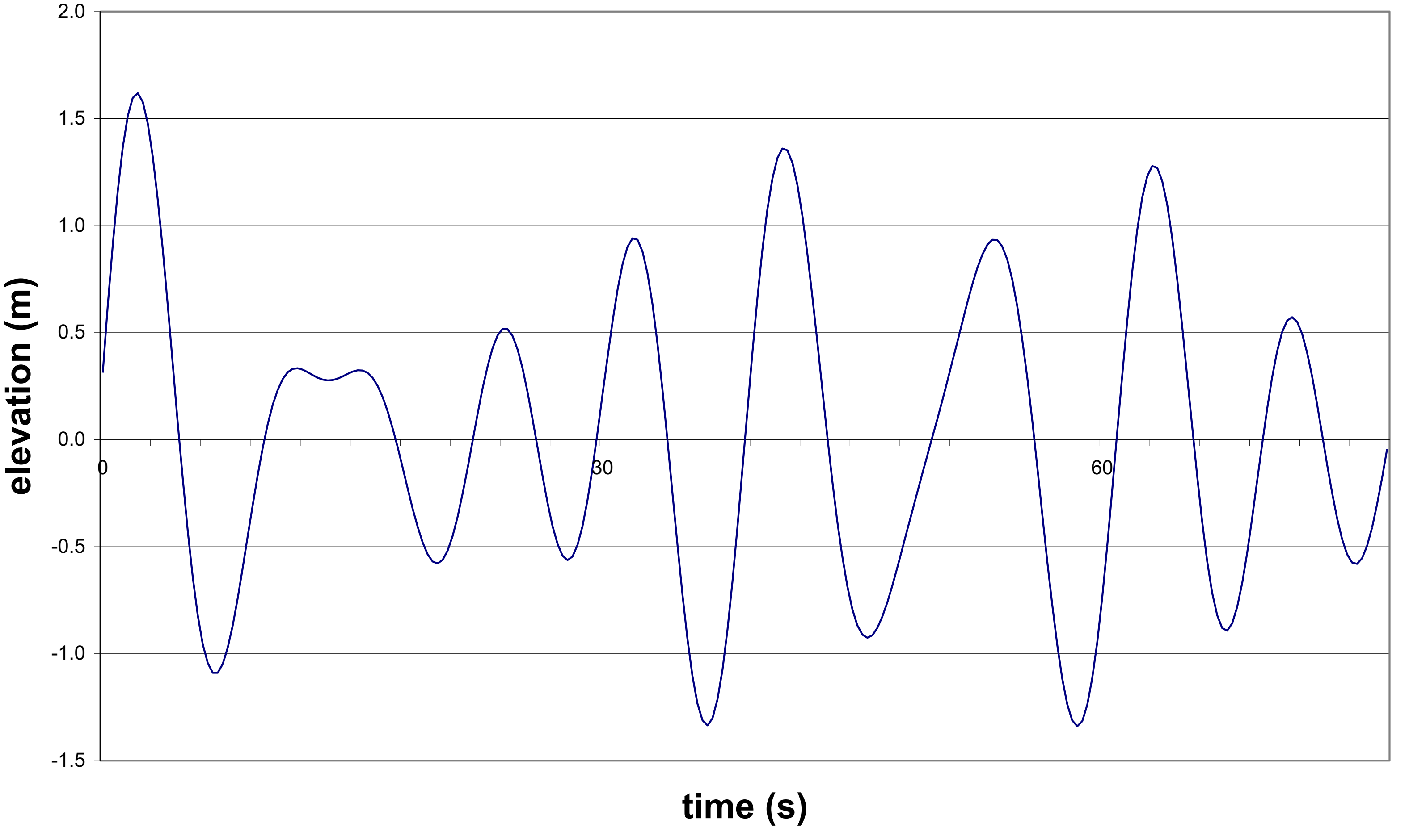
For linear waves, the combined free surface profile is the sum of the individual components

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3$$

# Realistic ocean surface from three sine waves



# Realistic ocean surface from three sine waves



# Summary

- Characteristic heights and periods for irregular waves are more complicated than for regular waves.
- Significant wave height is a useful height measurement.
- An irregular wave pattern can be created by summing a series of sine waves.

# 2. Irregular waves summary

You should now be able to:

- Develop an irregular wave time series from regular waves.
- Distinguish between the different methods of describing irregular wave characteristics.

# Real ocean waves content

1. Irregular waves

   Superposition, significant wave height

2. Wave spectra

   Spectral statistics, stylised spectra, encounter spectra, broadness, directionality

3. Probability distributions and extreme waves

   Rayleigh and Gaussian distributions, extreme wave height probability, joint probabilities

# 3a. Wave Spectra: what they are and how to use them

By the end of this section you should be able to:

- Explain how a wave spectrum is created.
- Apply a wave spectrum to a RAO to get the ship response spectrum.

# Wave spectra

- We have shown that a realistic wave record can be built by adding a series of sine waves. It should perhaps be possible to reverse the process and describe a real wave recording in terms of its sinusoidal components. This reverse process is called Fourier transformation, and leads to the concept of a wave spectrum.
- (Caution: not all processes are reversible like this e.g. when we look at directional wave spectra later).

# Wave spectra

The energy per square metre of ocean surface in a sinusoidal wave is

$$E = \frac{\rho g \zeta_a^2}{2}$$

$\zeta_a$ = wave amplitude

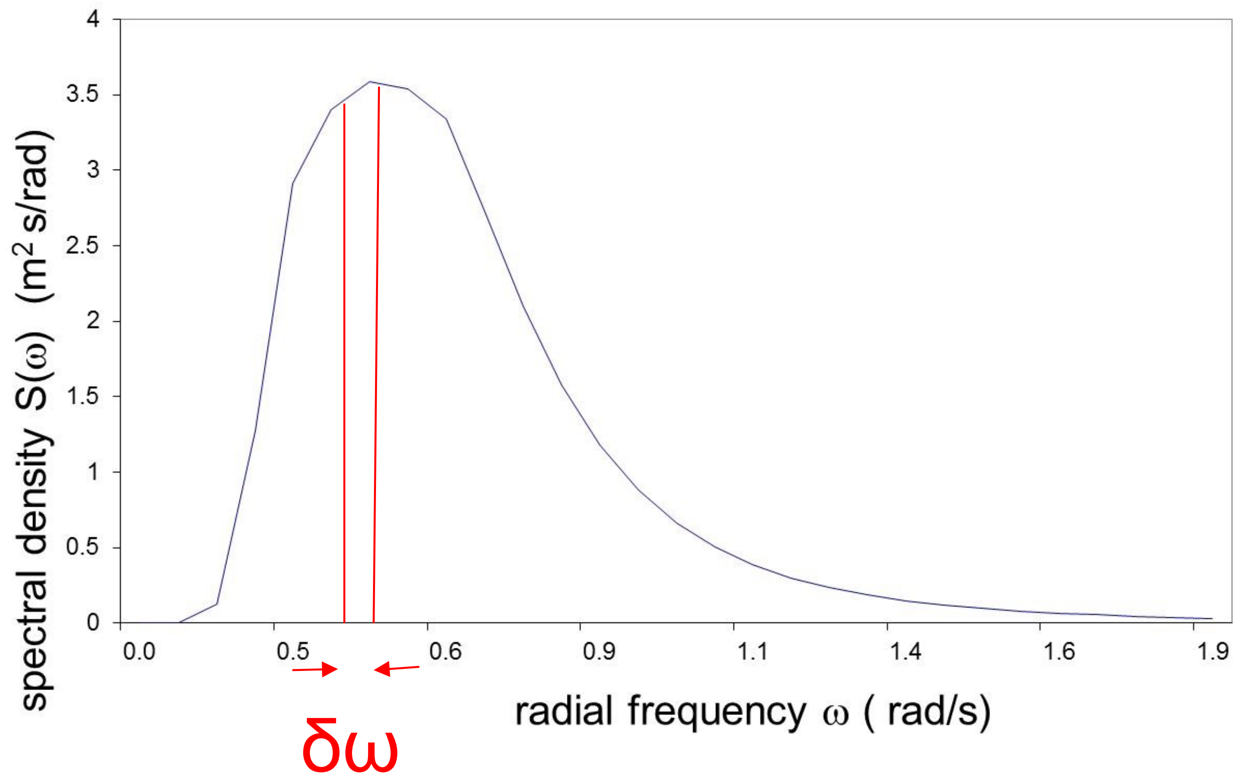
- So the total energy per $m^2$ for n waves is

$$E_{tot} = \frac{\rho g}{2} \sum_{i=1}^{n} \zeta_{a_i}^2$$

- Thus any given ocean wave field can be described by plotting the energy contained at each wave frequency present in that wave field. Such a plot is call an *energy density spectrum,* often shortened to just *wave spectrum*

( note: spectrum = singular, spectra = plural)

# Wave spectra



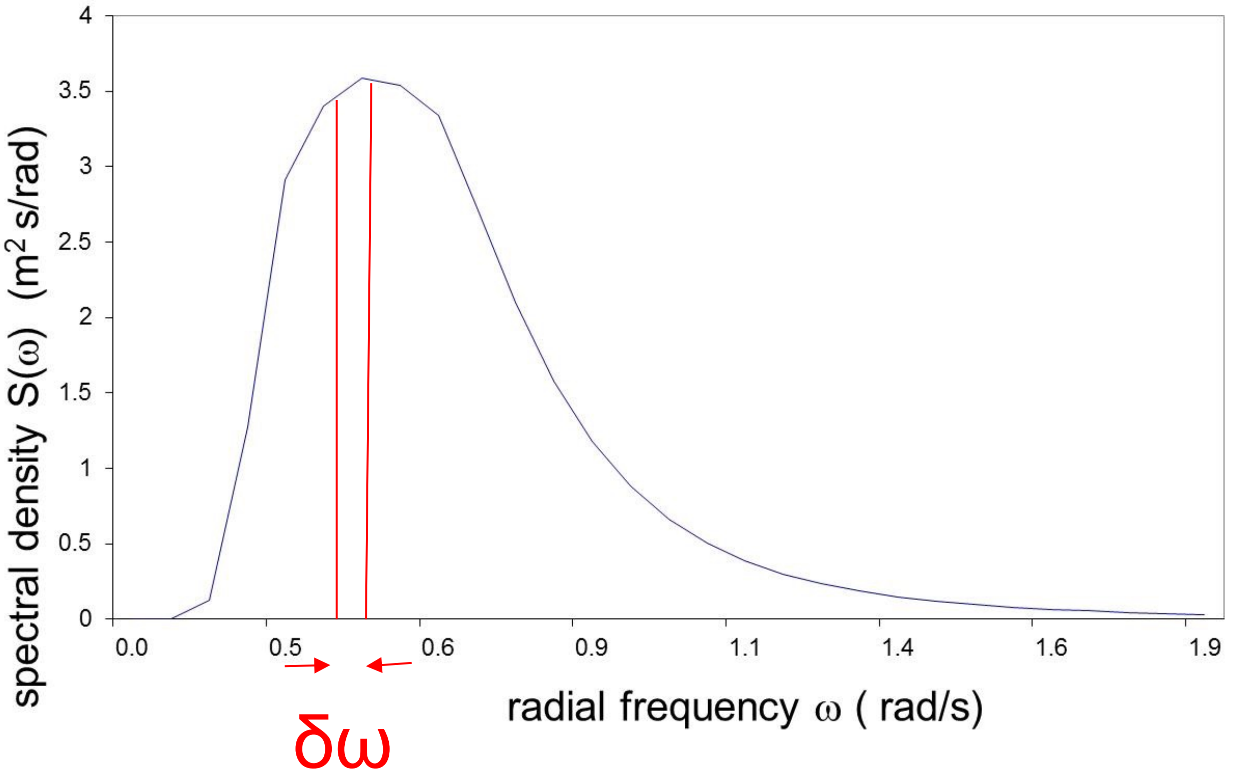

- The spectrum is plotted such that the area under a part of the curve $\delta\omega$ wide represents the energy per $m^2$ of sea surface for waves in that frequency band
- So total area under curve is proportional to the total energy per $m^2$ for that particular sea state

# Wave spectra and Fourier transforms

Recap your notes on Fourier transforms, especially the keywords Fast Fourier Transforms ( FFTs), Nyquist frequency, sample frequency.

- If a FFT is applied to a time series it generates a power spectrum. This is very similar to the wave spectrum just described, but there are some important differences
- One of the most important differences is the units used for the vertical ordinate. For an FFT, the size of the vertical ordinate varies with the sample rate selected (hence the frequency bandwidth), so the area under the spectrum would also vary with selected sample rate
- This would not work for a wave spectrum. Why not?

# Wave spectra and Fourier transforms




$$\text{Energy} \propto \int S(\omega)\,d\omega$$

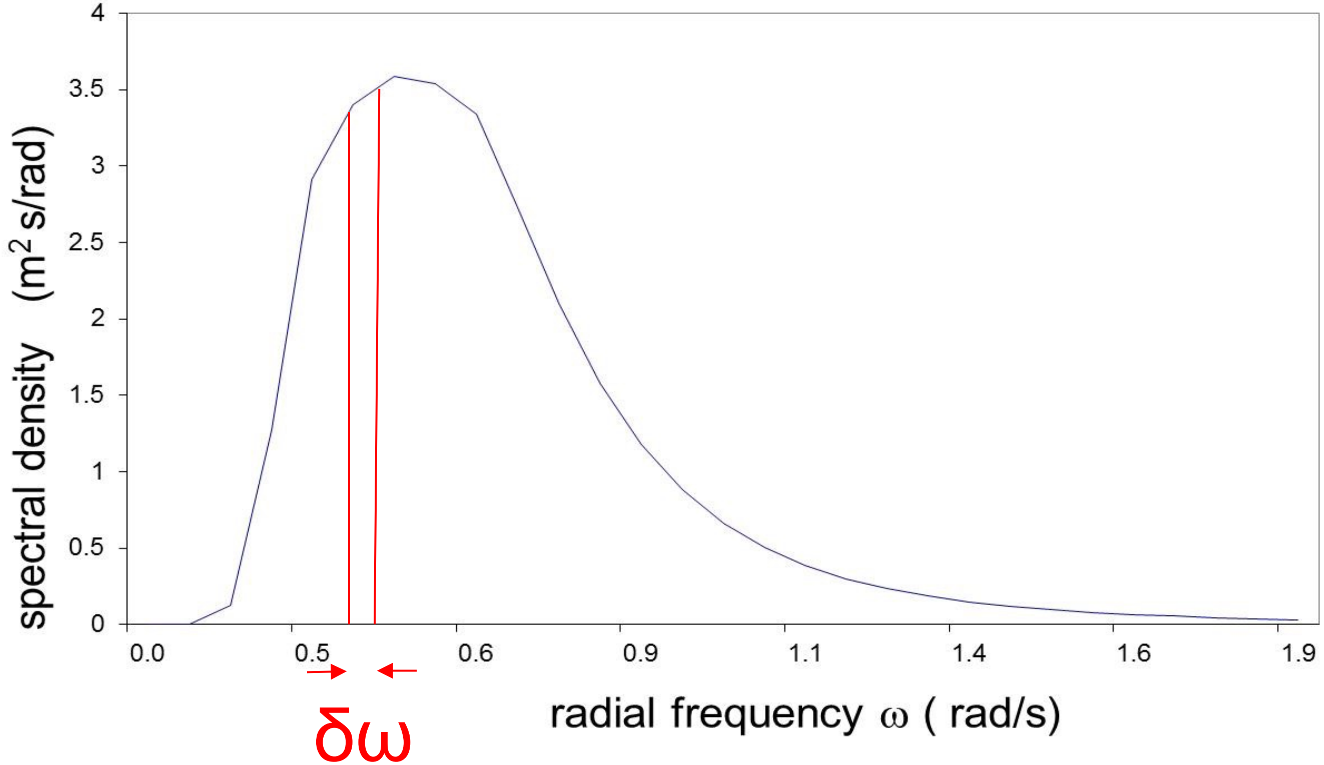
# Response of ship in an irregular seaway

- In an irregular seaway, a ship will have irregular oscillations.
- The pattern of these oscillations follows the same laws as the pattern of wave amplitudes.
- The histogram of wave amplitudes and ship motions approximately follows a Rayleigh distribution.

# Response spectra

- A wave spectrum is related to the wave amplitudes at each wave frequency.
- A response spectrum is related to the motion amplitudes at each wave frequency.

# Response spectra




- The wave spectral density $S(\omega)$ is related to the wave amplitudes in each frequency band.
- The response spectral density $S_z(\omega)$ is related to the motion amplitudes in each frequency band.

# Calculating motion properties from response spectra

Motion response can be calculated from the spectrum, e.g.

Significant motion amplitude (narrow-banded spectra)

$$x_{i\,sig} = 2\sqrt{m_0}$$

where

$$m_0 = \int_0^\infty S_{x_i}(\omega)d\omega$$

= area under spectrum

# Calculating the response spectrum

The response spectrum can be calculated from the wave spectrum and the RAO, based on two assumptions:

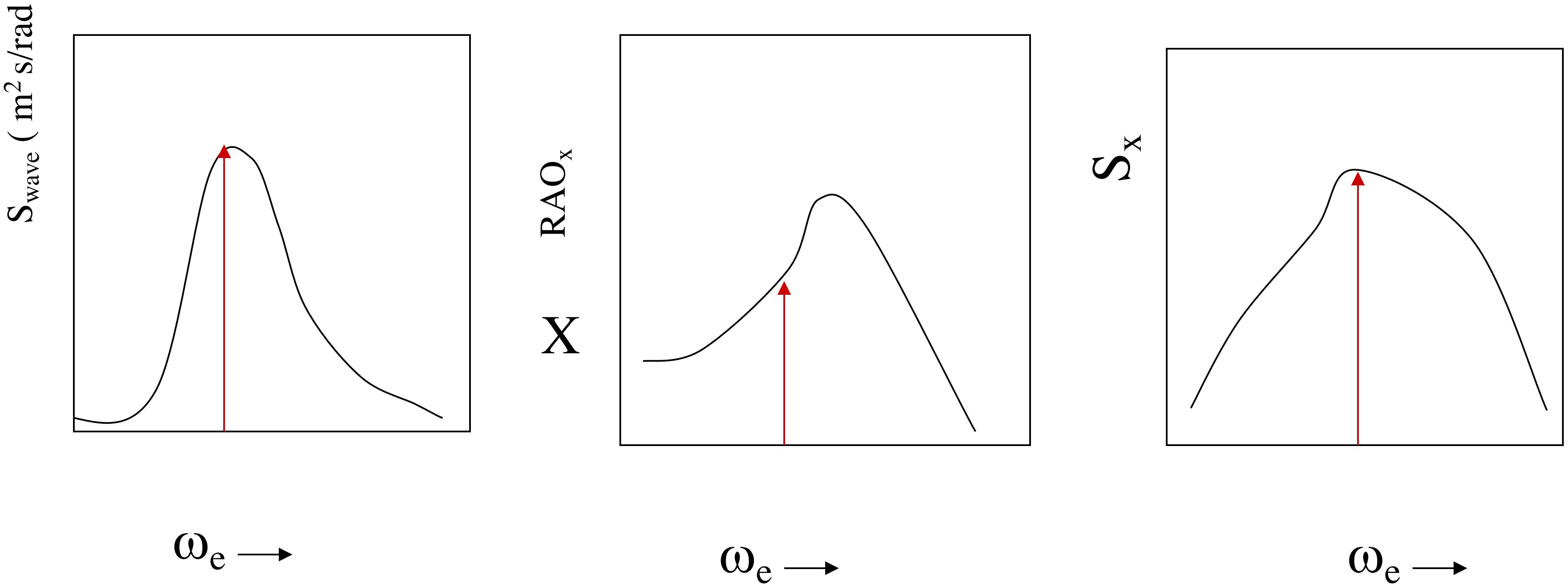
1. The response of the ship to any wave component is proportional to the amplitude of that component (linearity).
2. The response of the ship to any wave component is independent of its response to other wave components (superposition).

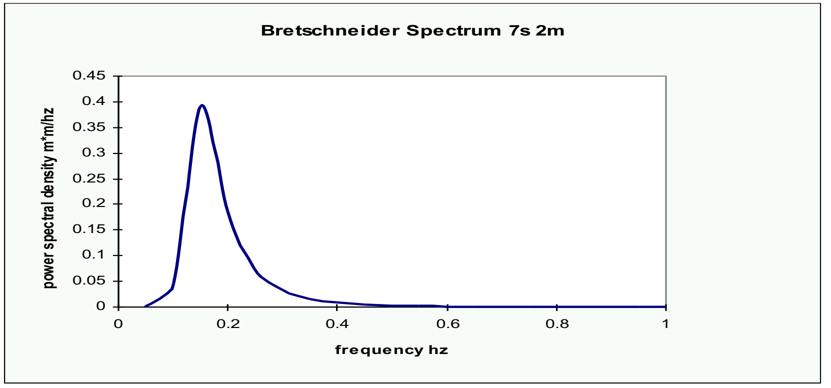
# Calculating the response spectrum of a stationary ship

In this case, the encountered wave spectrum is equal to the actual wave spectrum, and the response spectra can be calculated using:

$$S_{x_i}(\omega) = \mathrm{RAO}_{x_i}\ S(\omega)$$

# Calculating the response spectrum




Bretschneider Spectrum 7s 2m
power spectral density m*m/hz
0.45
0.4
0.35
0.3
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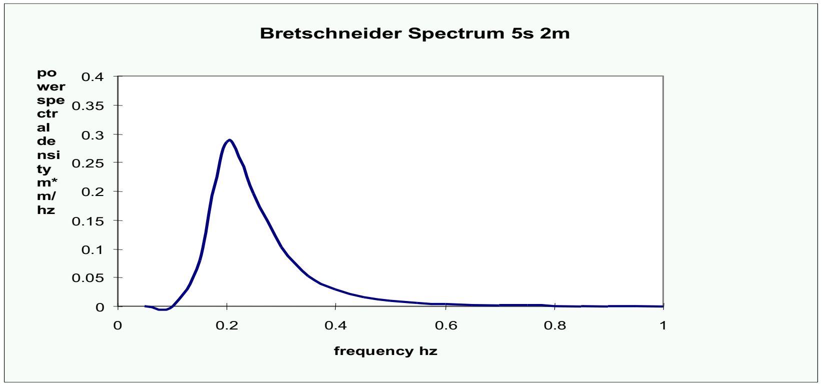
frequency hz


Bretschneider Spectrum 5s 2m
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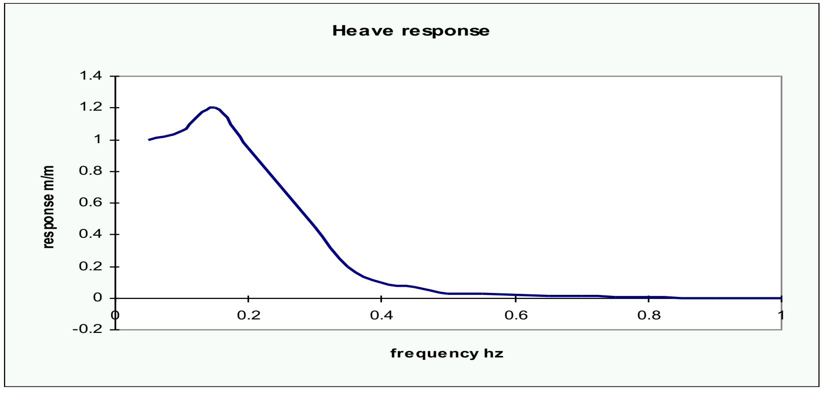
Heave response
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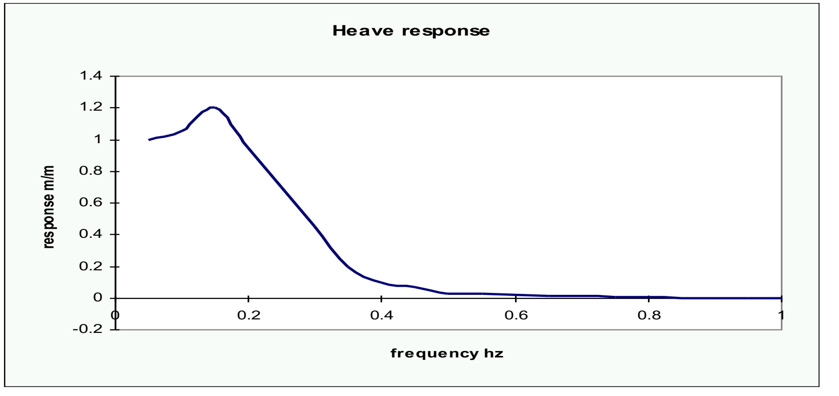
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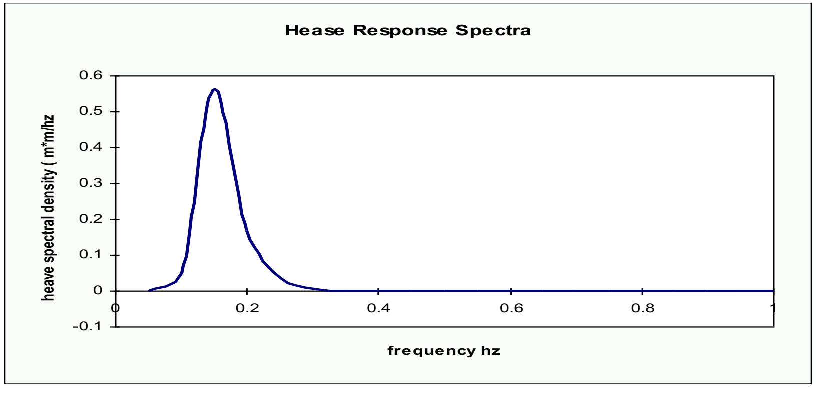
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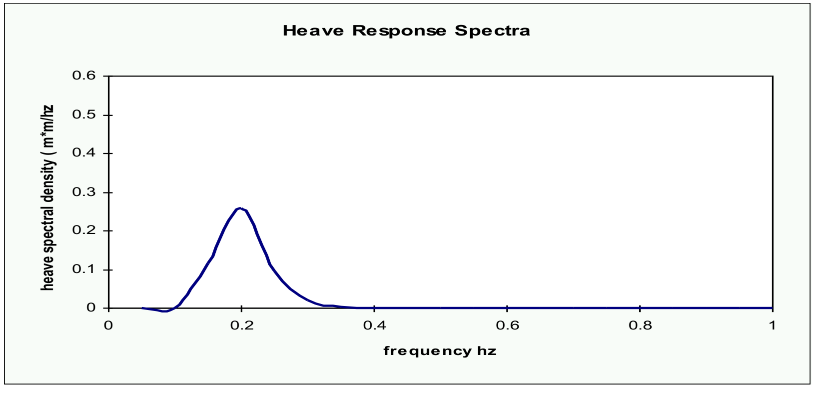
frequency hz


Heave Response Spectra
heave spectral density ( m*m/hz
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1
frequency hz

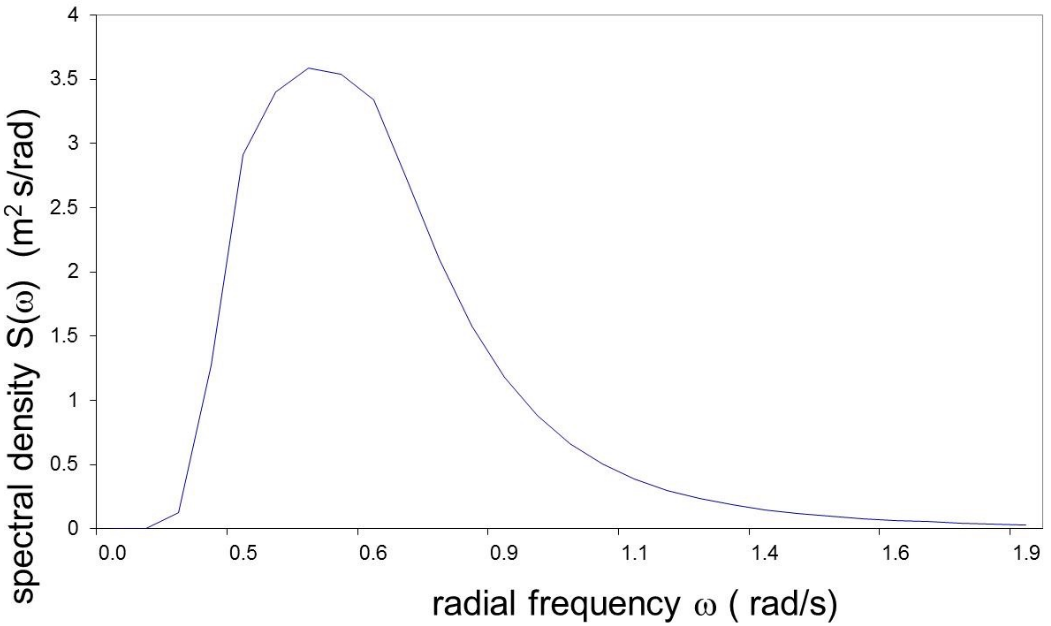
# 3b. Wave Spectra advanced
# Objectives

By the end of this section you should be able to:

- Describe how a wave spectrum is influenced by different factors.
- Calculate the statistical properties of a spectrum.
- Convert a wave spectrum to an encounter spectrum.
- Apply stylised spectra to given wind and sea conditions.
- Calculate a slope spectrum from an amplitude spectrum.
- Show the effect of spectral broadness on statistical properties.
- Explain how multidirectional seas are described spectrally.

# Effect of sea conditions on spectral shape

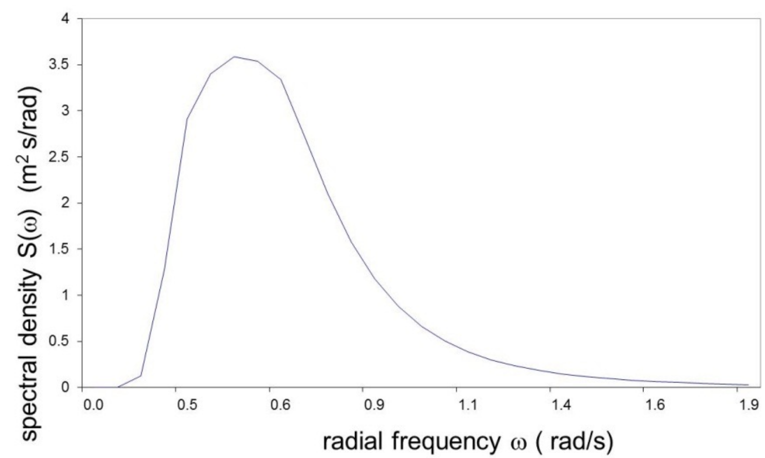
- Here is a typical open-ocean wave spectrum generated by a wind speed of 30 kn that has been blowing for one hour.




- If the wind continues to blow at the same speed, how would you expect it to change with time?

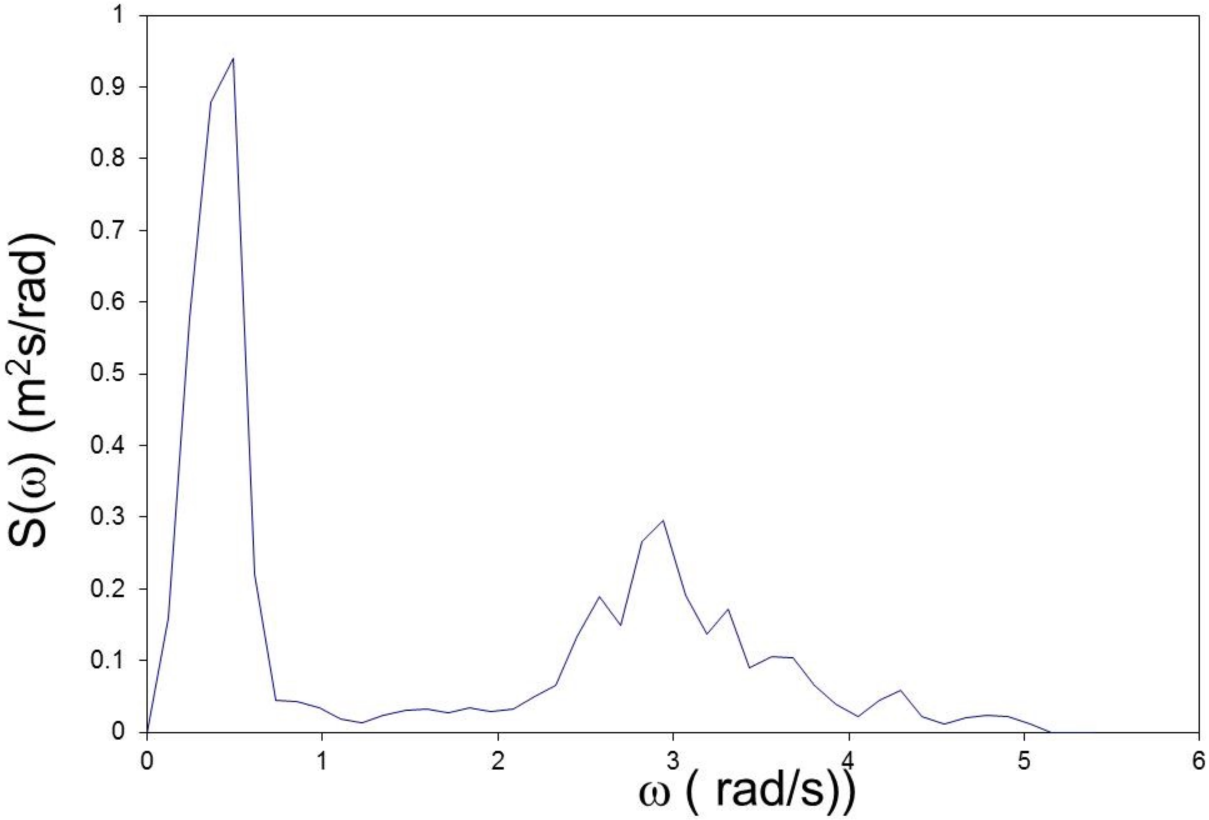
# Effect of sea conditions on spectral shape

- So far we have only considered an open-ocean wave spectrum. Often the wind is blowing away from a coastline shoreline, so the waves close to the coast have not had opportunity to develop.
- The distance over which the wind blows is called the fetch The spectrum below is generated by a wind speed of 25kn that has been blowing for 40 hours from off the shore. It was measured at a point 5km offshore ( i.e. fetch = 5km)




- If the wind continues to blow at the same speed, how would you expect the spectrum to vary as the measurement point is moved further offshore? Sketch your answer.

# Effect of sea conditions: double peaks




- What is happening here? There are two peaks in the spectrum; why?

# Statistical properties of a spectrum - height

- We have already covered the fact that the area under the spectrum is proportional to the energy (per $m^2$ of sea surface). It can be shown that the area under the spectrum is equal to the variance of the wave surface elevation and is given the symbol $m_0$. The significant wave height is related to both the variance and the spectral shape. For a narrow banded spectrum:

$$H_{sig} = 4\sqrt{m_0}$$

# Statistical properties of a spectrum - height

- However, ocean wave spectra vary in their narrow-bandedness (see later in this section), and the factor 4 varies accordingly. This causes confusion so, rather then using $H_{sig}$ as the characteristic wave height, the unambiguous term $H_{m0}$ is defined:

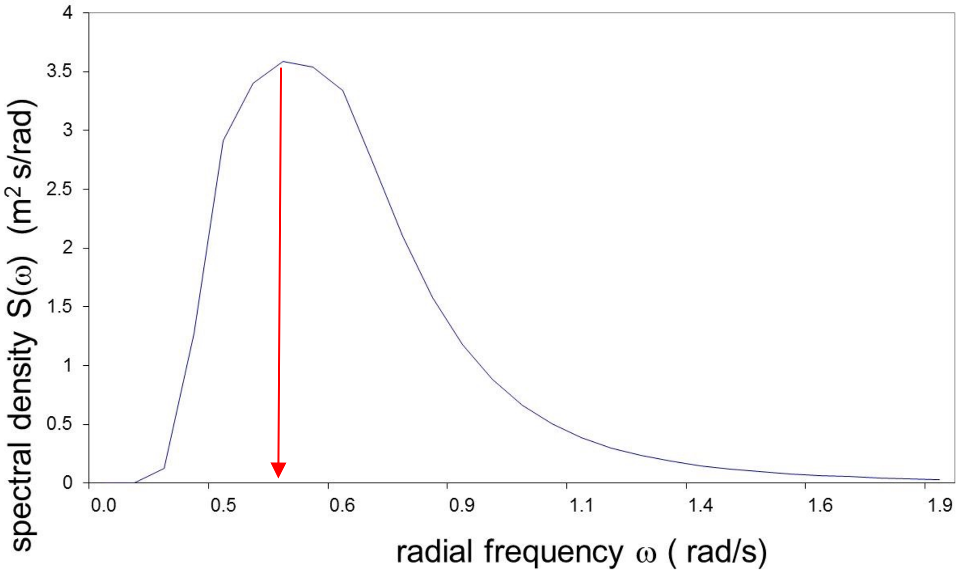
$$H_{m0} = 4\sqrt{m_0}$$

- This is very close in magnitude to $H_{sig}$ for most spectra.

# Statistical properties of a spectrum - period

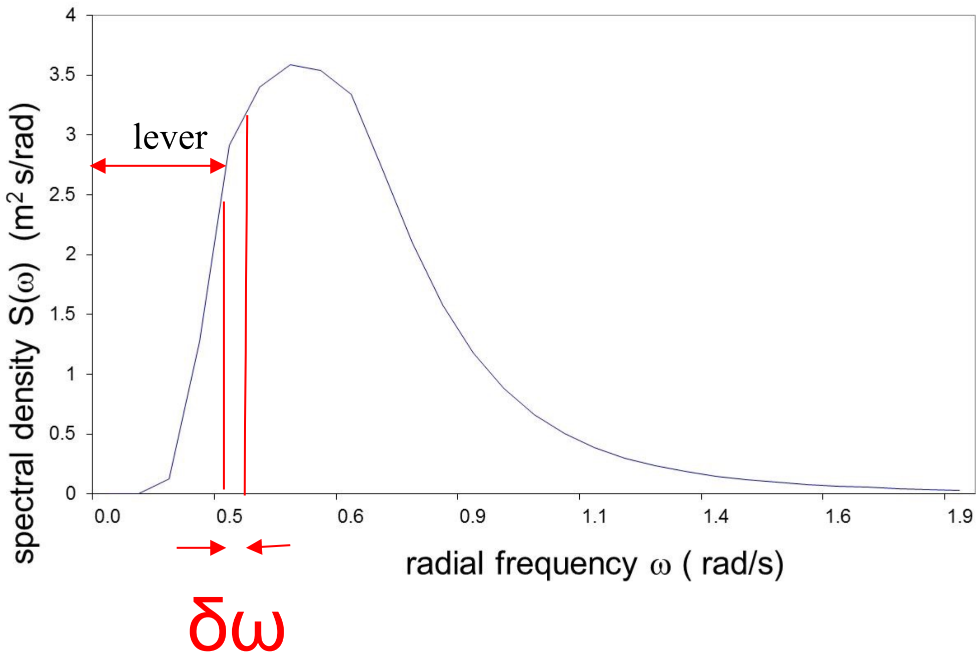
We have already seen that there are numerous characteristic wave periods (mean, modal etc.). These can be determined from the spectral shape – specifically from the frequency distribution.

The simplest and most obvious wave period for describing the spectrum is the modal or peak period $T_p$. This is the period which has the highest spectral amplitude.



# Statistical properties of a spectrum - period

- The other wave periods require more involved calculation of the spectral shape.
- This is done by taking moments of area about the vertical axis (similar to calculating moments of inertia for structural sections) . The spectrum is divided into segments or strips, the moment of each strip is calculated and the results for all strips is summed.



# Statistical properties of a spectrum - period

- The first moment of area of a strip is simply the area times the lever. The sum of these first moments is given the symbol $m_1$.
- The second moment is the area times the lever squared, the sum being given the symbol $m_2$. The third moment ( $m_3$) uses lever cubed etc.
- The most often used period is $T_{02}$ or $T_z$ , the zero-crossing period

$$T_z = \sqrt{\frac{m_0}{m_2}} \text{ for spectrum based on S(f) and f}$$

$$T_z = 2\pi\sqrt{\frac{m_0}{m_2}} \text{ for spectrum based on } S(\omega) \text{ and } \omega$$

- Sometimes the mean period $T_{01}$ is used

$$T_{01} = \frac{m_0}{m_1} \text{ for spectrum based on S(f) and f}$$

$$T_{01} = 2\pi\frac{m_0}{m_1} \text{ for spectrum based on } S(\omega) \text{ and } \omega$$

- The answer obtained depends on how far to the right (high frequency end) you integrate to.

# Same waves, different periods:

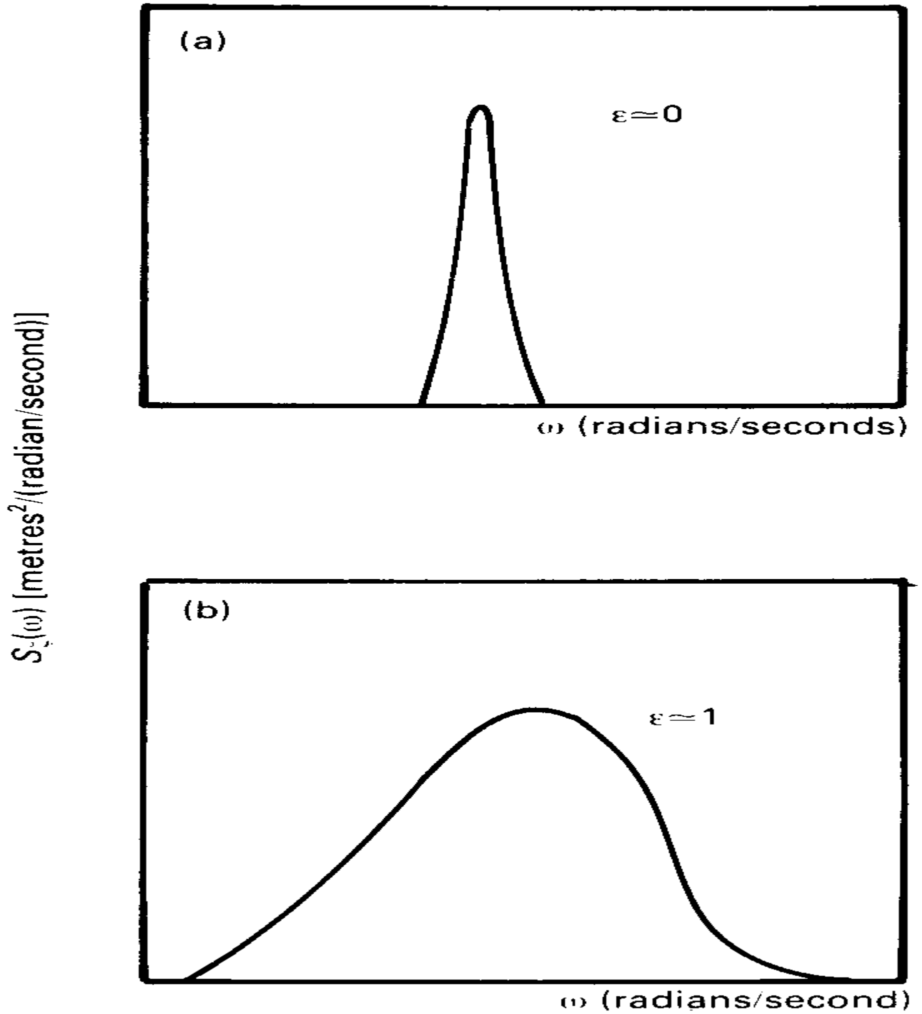
- To show why it is important to specify which characteristic period is being used, here are some values for a fully developed spectrum for 40kn wind speed:
- $T_z = 10.4$s
- $T_{01} = 11.3$s
- $T_p = 14.6$s

- Big differences!

# Summary

- A wave spectrum shows the energy distribution in a wave field.
- changes in environmental conditions affect spectral shape.
- statistical properties of wave height and period depend on the area under wave spectrum and associated moments.

# Spectral broadness:
# a) narrow b) broad



# Spectral broadness

- All the formulae given so far assume the spectrum is *narrow-band* i.e. the energy is concentrated in just a small frequency range. This is approximately true for most spectra, but not exactly.

- Why does this matter? It affects the statistical properties of the spectrum ($T_z$, $H_{sig}$ etc.)

# Spectral broadness

- There are two measures of broadness, both called "broadness parameter".
- ν is defined as:

$$\upsilon = \sqrt{\frac{m_0 m_2}{m_1^2} - 1}$$

$$m_i = \text{i}^{\text{th}} \text{ spectral moment}$$

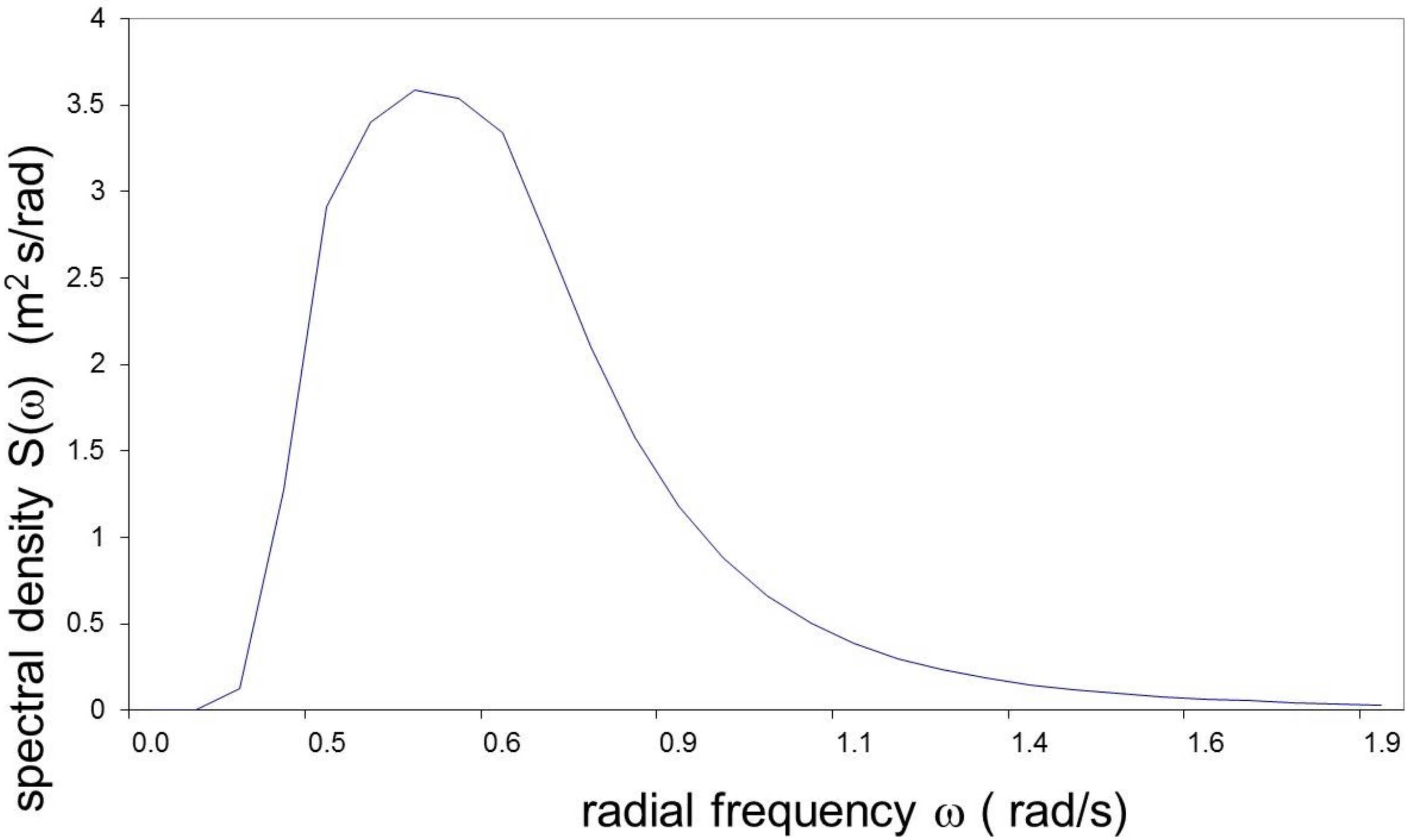
- and ε:

$$\varepsilon = \sqrt{1 - \left( \frac{m_2^2}{m_0 m_4} \right)}$$

# Spectral broadness

- In most information (especially before ~2000  and some RAN documents) ε is used. But it has a disadvantage….. What?

# Spectral broadness




- The right hand tail is small amplitude but very long, so value the $m_4$ is governed by what upper frequency the spectrum is cut off at.

# Why is broadness important?

- It alters all the relationships between the spectral moments and the characteristic heights and periods.

$$H_{sig} = \sqrt{\left(1 - \frac{\varepsilon^2}{2}\right)} H_{m0}$$

# Why is broadness important?

- ε varies from 0 for narrow-band spectrum to 1 for wide band spectrum so $H_{sig}$ varies from:

$$H_{sig} = 4\sqrt{m_0}$$

- To:

$$H_{sig} = 2.83\sqrt{m_0}$$

- i.e. 30% difference in $H_{sig}$ for the same value of $m_0$
- That is why we use $H_{m0}$ rather than $H_{sig}$
- Luckily $0.4<\varepsilon<0.8$ for most ocean spectra, which is a variation of less than 8% on $H_{sig}$

# © "Sea state"

| Sea state no. | $H_{sig}$ (m) range |
|---|---|
| 0-1 | 0–0.1 |
| 2 | 0.1-0.5 |
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| 7 | 6-9 |
| 8 | 9-14 |

Ref: World Meteorological Organisation

Ambiguous:

- Large range
- $H_{sig}$ or $H_{mo}$?
- What about period?

# (d) Standard or stylised spectra

- Quite often there is no measured wave data for the area of interest, or it is too expensive to carry out such measurements for preliminary design investigations. Fortunately a number of formulae have been developed which define the spectral curve as a function of various wave-influencing parameters.
- The simplest is the Pierson Moskowitz spectrum for fully developed seas in deep water, with wind speed the only variable:

$$S(\omega)=\frac{8.1\times10^{-3}g^{2}}{\omega^{5}}\exp\left[-0.74\left(\frac{g}{\omega V_{w}}\right)^{4}\right]$$

- Where S(ω) is the spectral ordinate ($m^2$ s/rad)
- $V_w$ is wind speed (m/s)
- ω is wave frequency (rad/s)

# Bretschneider spectrum

- The Bretschneider (or ITTC) “two parameter” spectrum assumes the sea is fully developed in deep ocean water

$$S_{B\zeta}(\omega) = \frac{A}{\omega^5} \exp\left[\frac{-B}{\omega^4}\right]$$

- where S = spectral ordinate ($m^2$ s/rad)
- $\omega$ = wave frequency (rad/s)
- To use this formula you have to calculate the values of the two parameters A and B by choosing a period and height:

# Bretschneider spectrum

$$S_{B\zeta}(\omega) = \frac{A}{\omega^5} \exp\left[\frac{-B}{\omega^4}\right]$$

$$A = 172.75 \frac{H_1^2}{T^4}$$

$$B = \frac{691}{T^4}$$

- $H_1$ is a characteristic height, usually $H_{m0}$
- T is a characteristic period, usually 1.09 $T_z$

Use this spectrum to determine spectral shape if you know the characteristic height and period

# JONSWAP (fetch-limited) spectrum

- The JONSWAP spectrum assumes the fetch is limited

$$S_{J\zeta}(\omega) = 0.658 C S_{B\zeta}(\omega)$$

- where $S_J$ = Jonswap spectral ordinate ($m^2$ s/rad)
- $S_B$ = Bretschneider spectral ordinate ($m^2$s/rad)

$$C = 3.3^{\exp\left[\frac{1}{2\gamma^2}\left(\frac{\omega T_p}{2\pi} - 1\right)^2\right]}$$

- Where $\gamma = 0.07$ for $\omega \le \frac{2\pi}{T_p}$
- Where $\gamma = 0.09$ for $\omega \ge \frac{2\pi}{T_p}$

# Seakeeping specification

The very weak statement:

*‘ The vessel must not roll more than 10 degrees in sea state 5’*

can now be improved e.g.

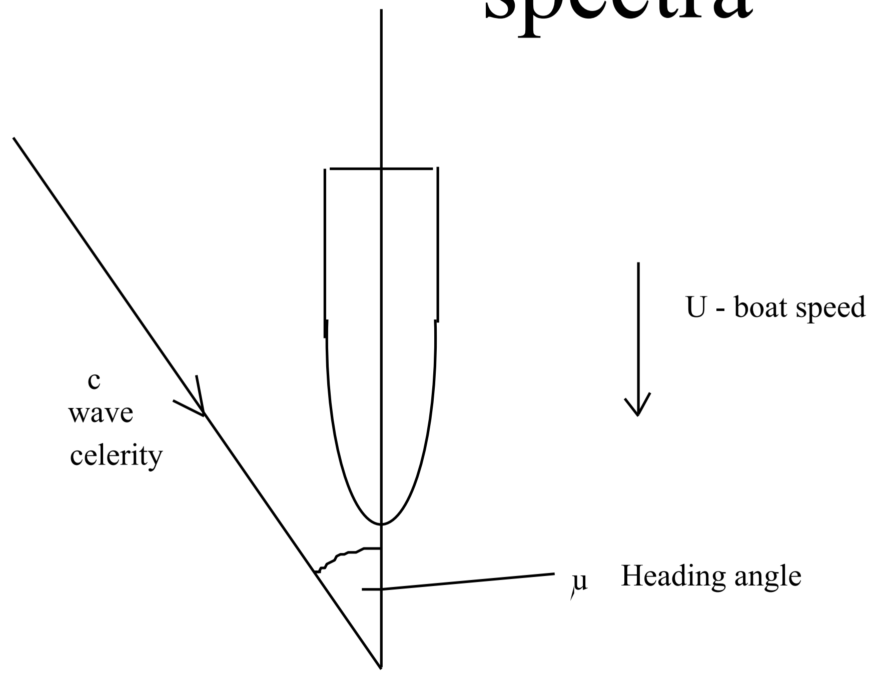
*‘...shall be fully operational in 4.0m Significant Wave Height (top of Sea State 5) and all Wave Zero Crossing Periods in the range of 6.0 to 12.0 seconds. A Bretschneider sea spectrum and long crested seas (zero spreading) shall be applied in demonstrating this requirement.’*

A016464 Standard Materiel Requirements for RAN Ships and Submarines
Volume 3: Hull System Requirements, Part 6: Seakeeping

# Encounter frequency and encounter spectra

- So far we have considered the wave field measured at, or experienced at a single point.
- If we are measuring or observing it from a moving object e.g. a ship, there will be a Doppler shift of frequency i.e. the frequency encountered will be different from the wave frequency.
- The encounter frequency will depend on the ship speed and direction of travel (heading) relative to the waves.

# Encounter frequency and encounter spectra




$\mu = 0°$ is a following sea

$\mu = 180°$ is a head sea

$\mu = 90°$ is a beam sea

$\mu = 0\text{-}90°$ or $270\text{-}360°$ is a stern quartering sea

$\mu = 90\text{-}270$ is a bow quartering sea

# Encounter frequency

- Consider waves of frequency ω, wave length λ travelling at celerity (phase velocity) $C_w$. The velocity of the wave relative to the ship, travelling at speed U, is then:

$$C_w - U\cos(\mu)$$

- The time between encountering wave crests ( the encounter Period $T_e$) is then

$$T_e = \frac{\lambda}{C_w - U\cos(\mu)}$$

# Encounter frequency

- So the encounter frequency is:

$$\omega_e = \frac{2\pi}{T_e} = \frac{2\pi\left(C_w - U\cos(\mu)\right)}{\lambda}$$

- Now

$$C_w = \frac{\omega\lambda}{2\pi}$$

# Encounter frequency

And for deep water waves

$$\lambda = \frac{gT^2}{2\pi}$$

- So

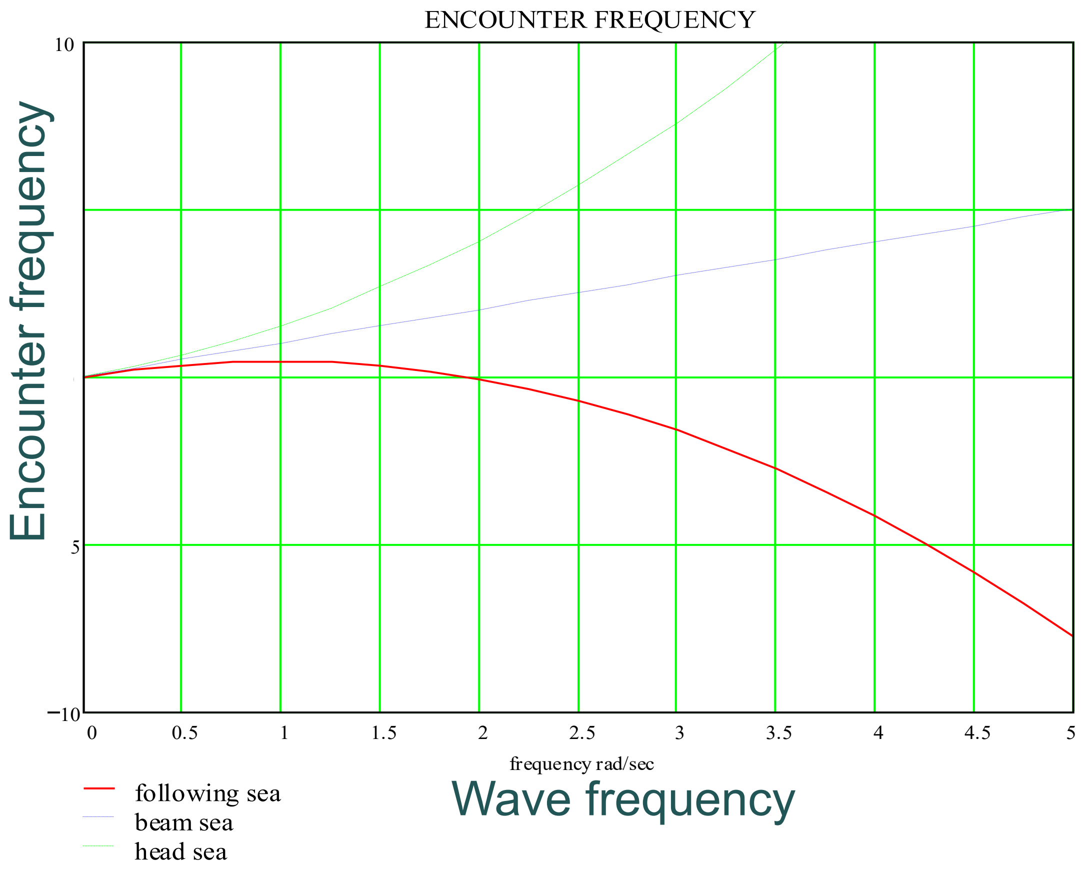
$$\frac{\omega^2}{g} = \frac{2\pi}{\lambda}$$

- Then……..

$$\omega_e = \omega - \frac{U\omega^2}{g}\cos(\mu)$$

# Encounter frequency

- The graph below illustrates three cases for a vessel travelling at 5m/s (approx. 10kn)



# Encounter spectra

- We now have the formula for calculating the frequency of the wave as experienced from a moving object. This means that the horizontal axis of a wave spectrum will change if it is measured or experienced from a ship. This will change the area under the spectrum.
- But the area under the spectrum represents the wave energy, and this does not change just because we are observing it from a moving object.
- What must be done to the shape of the encountered spectrum to keep the area unchanged?

# Encounter spectra

- The vertical ordinates must be adjust in proportion to the change in the horizontal ordinates. How?
- The area of a strip of the wave spectrum must be the same as the area of the equivalent strip of the encounter spectrum:

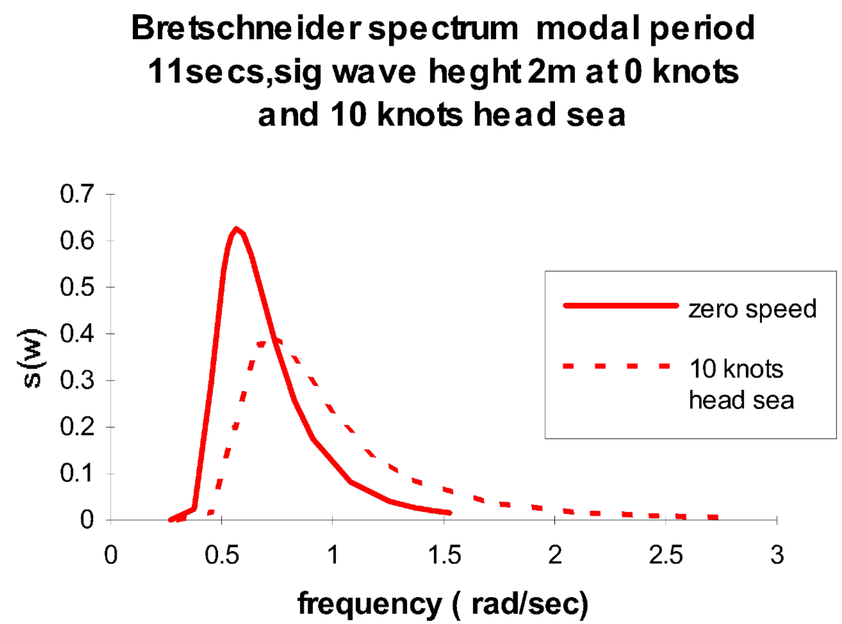
$$S(\omega_e)\delta\omega_e = S(\omega)\delta\omega$$

- Hence the new vertical ordinates are found from the differential of the encounter frequency formula already derived. Pause slide show and do the differentiation….

# Encounter spectra

$$S(\omega_e) = \frac{S(\omega)}{1 - \dfrac{2\omega U}{g}\cos\mu}$$

- The converted spectrum is called an encounter spectrum. It will look similar to the original wave spectrum. For a ship travelling into the waves (head seas), the peak will be reduced in height and shifted to the right (higher frequency)



# Wave slope spectra

- So far all the spectra we have described are amplitude or height spectra. Sometimes a spectrum of wave slopes is required. This is easily obtained from the wave amplitude spectrum by noting the formula for maximum slope of a sine wave:

$$slope = \frac{2\pi\zeta_a}{\lambda}$$

- Now for deep water:

$$\lambda = \frac{2\pi g}{\omega^2}$$

- So to obtain the slope spectrum you just multiply each amplitude spectral ordinate by

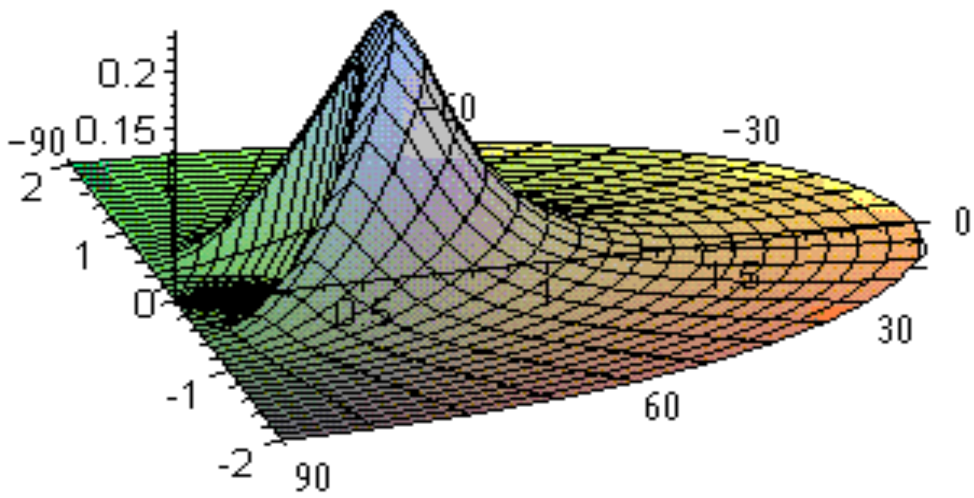
$$\frac{\omega^4}{g^2}$$

# Directional spectra

- So far we have assumed that the waves are all coming from the same direction. This assumption was not necessary for using any of the formulae for characteristic wave heights and periods, but it does affect how you get to these formulae and it introduces two other measures of the wave field – the modal direction and the spread.
- Real waves usually come from one dominant direction, but some of the waves are at slightly different directions, usually up to 90 deg either side of the main direction.
- It is unusual for waves to come from more than 90 deg, this would mean that some waves would be running in opposite direction to others. But it is possible.

# Directional spectra

- The energy is spread around different directions and our two-dimensional wave spectrum graph should really be a 3D one:



# Directional spectra

- If all the waves travel in one direction only, the system is called *one-dimensional*, or *long-crested.*
- If the waves come from different directions the wave system is called *two-dimensional* or *short-crested*.

# Directional spectra

- For multi-directional spectra the spectral density will be a function not only of frequency but also direction

$$S(\omega)=\int S(\omega,\theta)d\theta$$

- Where θ = direction of wave component relative to modal direction
- The area under the wave spectrum thus becomes

$$m_0=\int_0^\infty \int_{-\pi}^{\pi} S(\omega)D_x(\theta)d\theta d\omega$$

- Where $D_x$ is called the *spreading function.*
- It varies with wave angle and also wave frequency, though the latter is not always included.

# Directional spectra

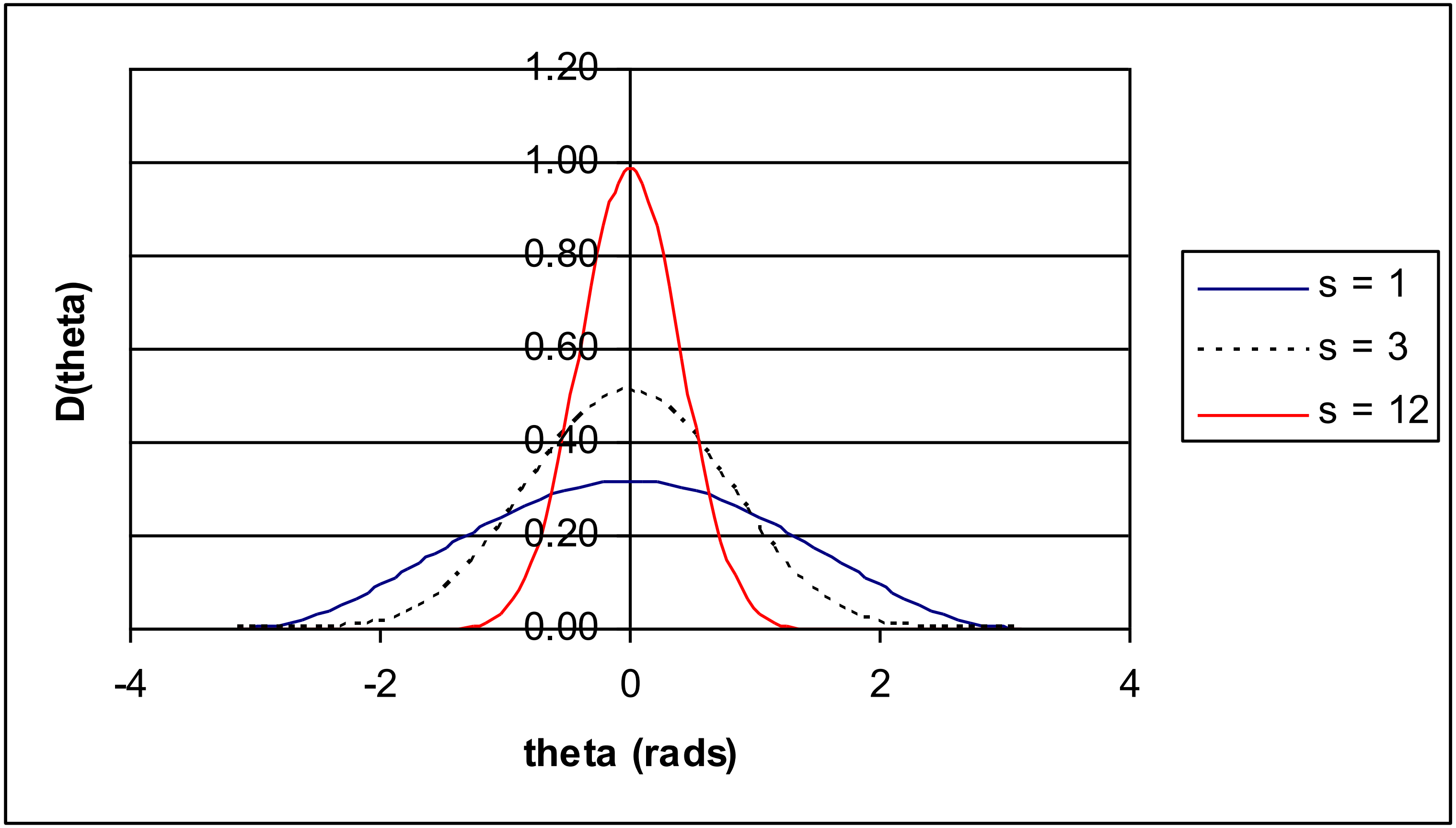
- The spreading function is a measure of how much the energy is spread around the dominant direction.
- Since the directionality of the spectrum cannot alter the total energy and the total energy must still be represented by the total area under the spectrum, the integral of the spreading function must be 1.0. The most commonly used spreading function is

$$D_{xs}(\theta) = D_{x(s-1)}\left(\frac{s}{(s-0.5)}\right)\cos^{2s}\left(\frac{\theta}{2}\right)$$

where $D_{x1}(\theta) = \frac{1}{\pi}$

- Check for yourself that the integral of this equation from minus 180 deg to plus 180 deg is equal to 1, regardless of the value of s.

# Directional spectra




- A typical value of s is 12. High value = low spread, low value = large spread

# Recap - calculating wave properties from wave spectra

- Some wave properties that can be calculated from wave spectra are as follows:

  First moment of area

$$m_0 = \int_0^\infty S(\omega)d\omega$$

  Significant wave height (narrow-banded spectra)

$$H_{sig} = 4\sqrt{m_0}$$

  Significant wave amplitude (narrow-banded spectra)

$$a_{sig} = 2\sqrt{m_0}$$

# Wave spectra and Fourier transforms

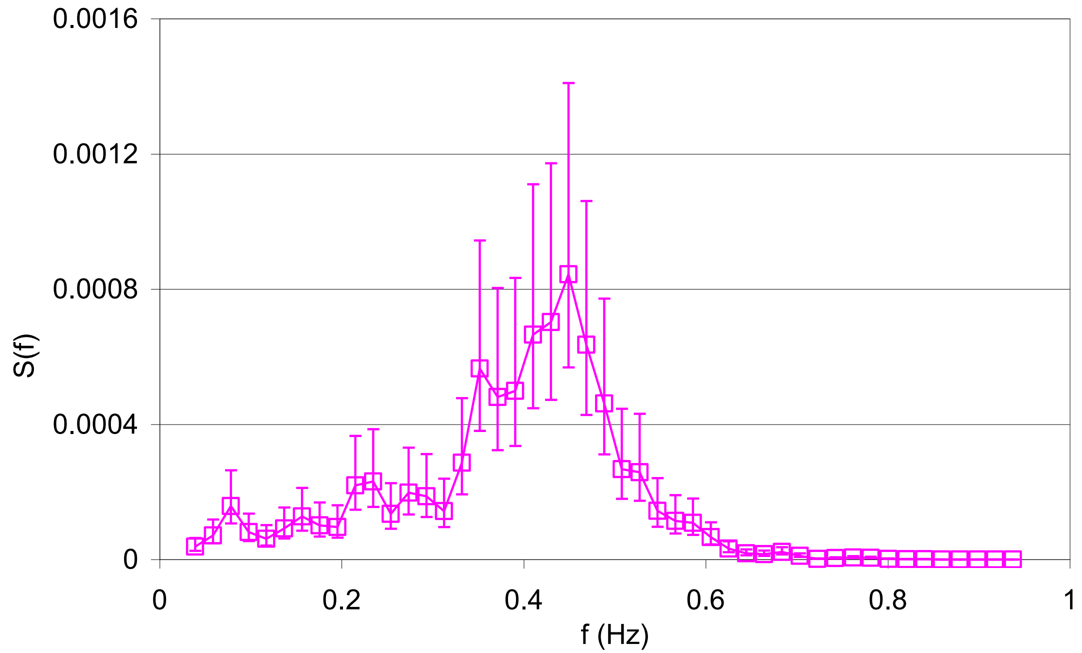
- There are several other factors to be aware of when applying an FFT to a time series to obtain a wave spectrum. The following affect the magnitude of the ordinates, but not the shape of the spectrum:

1. Double-sided spectra (folded about the Nyquist frequency)
2. Imaginary component of the spectrum
3. Amplitude, height or energy ordinate?

- Other processing factors also affect the spectral shape e.g. windowing, record length, overlapping. These will be discussed under Full Scale Trials Analysis.

# A note on uncertainties in spectra

- Users of recorded wave spectra often assume the data is precise and accurate. They forget they are looking at a statistical sample, with inherent uncertainties. These uncertainties depend not only on the original time series, but also on the way the data has been processed.
- 90% confidence limits can be as much as 30% of the spectral ordinate value.



# Summary

- Stylised spectra can be used when full wave records are not available.
- A slope spectrum can be calculated from an amplitude spectrum.
- For an object travelling in waves the frequency of encounter and the encountered spectrum will change with speed and direction of travel.
- Multidirectional seas are described using a spreading function.

# 3. Wave Spectra summary

You should now be able to:

- Explain how a wave spectrum is generated.
- Describe how a wave spectrum is influenced by different factors.
- Calculate the statistical properties of a spectrum.
- Convert a wave spectrum to an encounter spectrum.
- Apply stylised spectra to given wind and sea conditions.
- Calculate a slope spectrum from an amplitude spectrum.
- Show the effect of spectral broadness on statistical properties.
- Explain how multidirectional seas are described spectrally.

# Real ocean waves
# content

1. Irregular waves

    Superposition, significant wave height

2. Wave spectra

    Spectral statistics, stylised spectra, encounter spectra, broadness, directionality

3. Probability distributions and extreme waves

    Rayleigh and Gaussian distributions, extreme wave height probability, joint probabilities

# Why conduct seakeeping analysis?

Determine the motions of a design in conditions it is likely to encounter

Predict extreme (worst) waves ever to be experienced

Predict worst waves to be experienced during (short) mission

- **Is the vessel going to survive?**
- **Can the vessel carry out specified task or mission?**
- Decide if motions are acceptable:

  Slamming, Deck Wetness, Speed Loss, Human Performance, Ride Control

- Decide which design is going to perform the best:

  Design selection, marketing

# 4. Probability distributions
# Objectives

By the end of this section you should be able to:

- Identify the need to predict short- and long-term wave probability and extreme wave events.
- Explain why probability density functions and error functions are used.
- Show why wave amplitudes have a Rayleigh distribution.
- Calculate exceedance probabilities and joint probabilities of height and period.
- Estimate extreme wave heights.

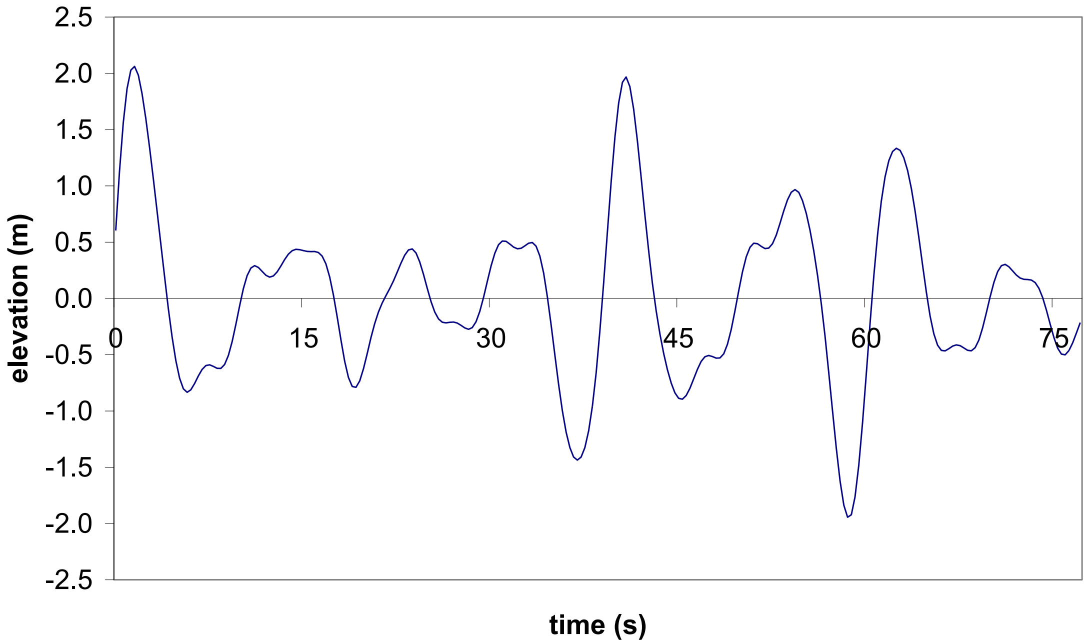
# Probability distributions of waves

Key phrases for revision: Rayleigh distribution; Gaussian distribution; probability density function (pdf); error function

- Failure of a ship structure due to waves can be due to either fatigue or excessive force from a single event (extreme wave).
- Fatigue analysis requires the designer to know the long-term joint probability of wave height, period ( and direction) of all the waves likely to be encountered.
- Single event failure design requires an estimate of the height, period (and direction) of the single extreme wave likely to be experienced within a specified time e.g. 20 or 30 years.
- Short duration operations (vessel transfers, helicopter landings etc.) require prediction of highest likely wave in a given short time period.
- All these predictions require an understanding of the probability distribution of ocean wave heights and periods.

# Probability distributions of waves

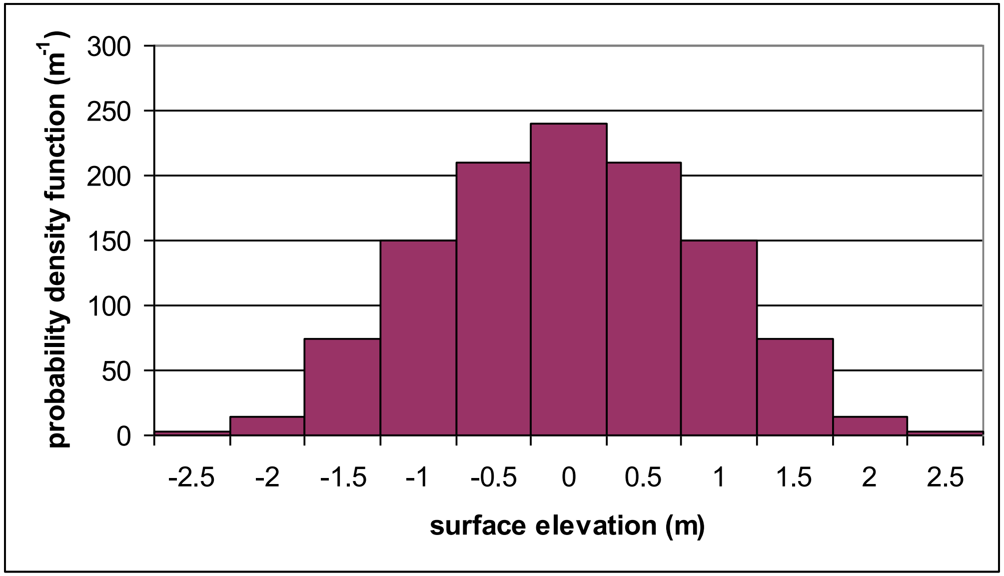
- Consider the sea surface elevation record below:




- If the record were sampled at a suitably small time interval, what would the probability distribution of those sampled values be – Gaussian, Rayleigh or something else?

# Probability distributions of sea surface elevation

The distribution would be Gaussian i.e. a "bell curve"




- The probability of a particular elevation is the same as the probability of the equivalent amplitude of depression, so the curve is symmetric about the mean, and the probability of a particular elevation decreases as the elevation gets higher.
- Note the vertical axis is not probability, it is probability density function. Why?

# Probability distributions of sea surface elevation

The pdf is defined such that the area enclosed by the pdf curve over a range of amplitude is equivalent to the probability of a measurement falling within that range.

The pdf ordinate f for a Gaussian distribution with zero mean is given by:

$$f = \frac{1}{\sqrt{2\pi m_0}} \exp\left( \frac{-\zeta^2}{2m_0} \right)$$

- Note: ***f*** is NOT frequency!

# Probability distributions of sea surface

- The probability that an individual measurement will lie within a range is given in terms of the error function erf() as

$$P(\zeta_1 < \zeta < \zeta_2) = erf\left(\frac{\zeta_2}{\sigma_0}\right) - erf\left(\frac{\zeta_1}{\sigma_0}\right)$$

- Where

$$x = \frac{\zeta_i}{\sigma_0}$$

$$z = \zeta$$

$$\sigma_0 = \text{standard deviation}$$

$$erf(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp\left(\frac{-z^2}{2}\right) dz$$

# Probability distributions of wave amplitude

- So far we have considered the distribution of the surface elevation, and found it is Gaussian.
- What about the distribution of wave *amplitudes* or *heights* - is that also Gaussian?

# Probability distributions of wave amplitude

Rayleigh distribution probability density function is:

$$f = \frac{\zeta_a}{m_0} \exp\left( \frac{-\zeta_a^2}{2m_0} \right)$$

$m_0$ = variance

$\zeta_a$ = amplitude

# Probability distributions of wave amplitude

And the probability P that a chosen amplitude will be exceeded is:

$$P(\zeta > \zeta_{a1}) = \exp\left(-\frac{\zeta_{a1}^2}{2m_0}\right)$$

$$m_0 = \text{variance}$$

$$\zeta_{\text{a1}} = \text{amplitude to be exceeded}$$

# Joint Probability distributions

We often need to know the probability of two events occurring simultaneously. If the events are independent, the joint probability is just the product of the two individual probabilities:

$$P\left(x_i > x_{i1} \text{ and } x_j > x_{j1}\right) = \exp\left(-\frac{x_{i1}^2}{2m_{0i}}\right)\exp\left(-\frac{x_{j1}^2}{2m_{0j}}\right)$$

$m_0 = \text{variance}$

$x_1 = \text{amplitude to be exceeded}$

# Joint Probability distributions

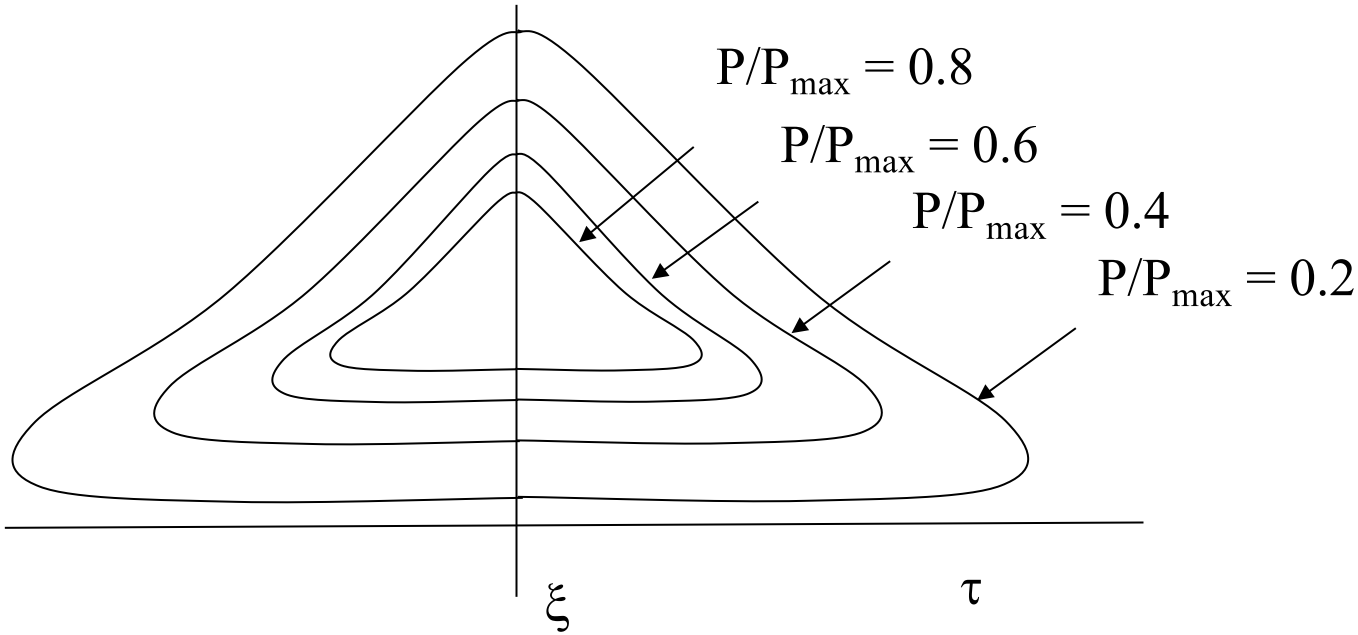
- Unfortunately, wave period and wave height are not independent; they are correlated.
- Theory of Longuet Higgins enables joint probability density to be calculated:

$$P(\xi,\tau)=\frac{\xi^2}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\xi^2\left(1+\tau^2\right)\right)$$

$\xi$ = dimensionless wave height

$\tau$ = dimensionless wave period

# Joint Probability distributions




$$P(\xi,\tau) = \frac{\xi^2}{\sqrt{2\pi}} \exp\left( -\frac{1}{2}\xi^2 \left(1+\tau^2\right)\right)$$

$$\xi = \text{dimensionless wave height} = \frac{H}{2.83\sqrt{m_0}}$$

$$\tau = \text{dimensionless wave period} = \frac{T - T_{01}}{\upsilon^2 T_{01}}$$

# Estimating maximum wave heights- short term

- It is often necessary to estimate the maximum wave height experienced over a few hours e.g. ship-ship transfers

- Once the likely spectrum is determined, the most probable largest height is

$$H_{\max} = 2\sqrt{m_0 2 \log_e n}$$

- Where n is the number of waves experienced
- Note this formula is correct for narrow-band spectra, it overestimates the height for wide band spectra

# Estimating extreme (long term) wave heights

- Structural design is often based on highest wave likely to be experienced within the life of the structure. The duration between such waves is usually called the return period e.g. 20 year return period.
- It is difficult to estimate this accurately, but very important to try e.g. for every 1% increase in design wave height, the cost of building the structure to withstand it increases by 1-3%.

# Estimating extreme wave heights: summary of method

- Collect/measure or hindcast the $H_{m0}$ for the location over a long period (3 years, typically).
- Plot the distribution of $H_{mo.}$
- Find a probability distribution that fits the data.
- Extend it to the return period (e.g 20 years).
- Assume a design risk probability (e.g. 0.01).
- Use the chosen probability distribution to find $H_{m0}$ design.
- Warning: this is an area of much research; methods will change over time as our understanding improves.

# Estimating extreme wave heights:

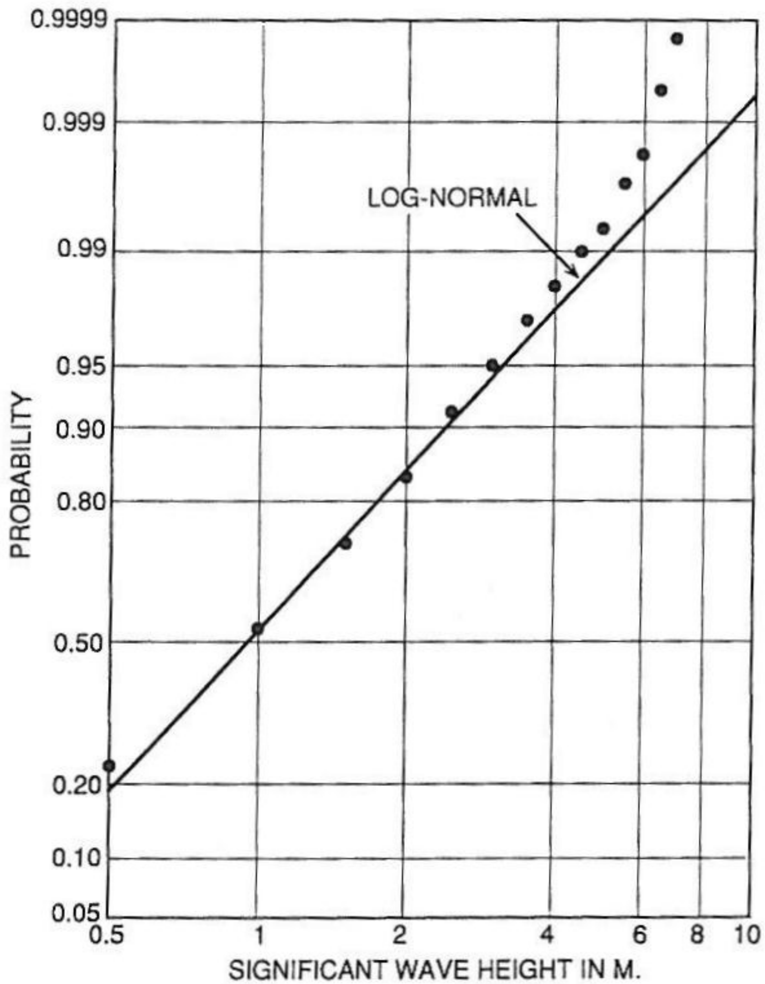
Highest likely wave is estimated from given wave climate which has some probability distribution.

No scientific basis for selecting the prob dist – just use one that best fits the data. Three options:

- log-normal- simplest, fits low wave heights well
- Weibull – fits high wave heights well
- generalised gamma – good across the range

# Log-normal distribution

Log-normal distribution fits the low height waves but not the extreme waves.

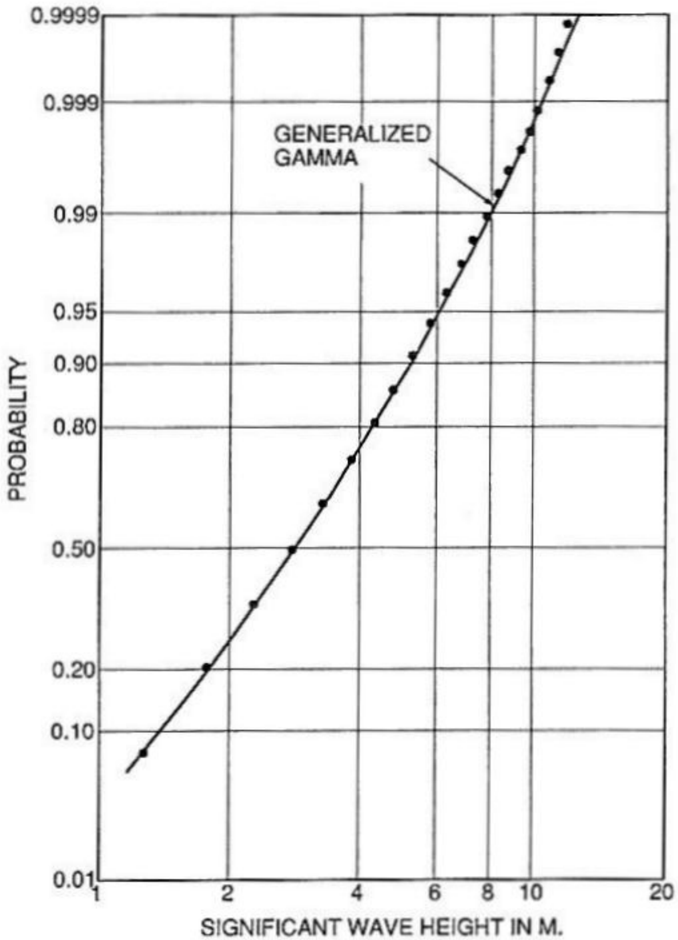



Tucker & Pitt (2001)

Ochi (1998)

# A better fit of distribution

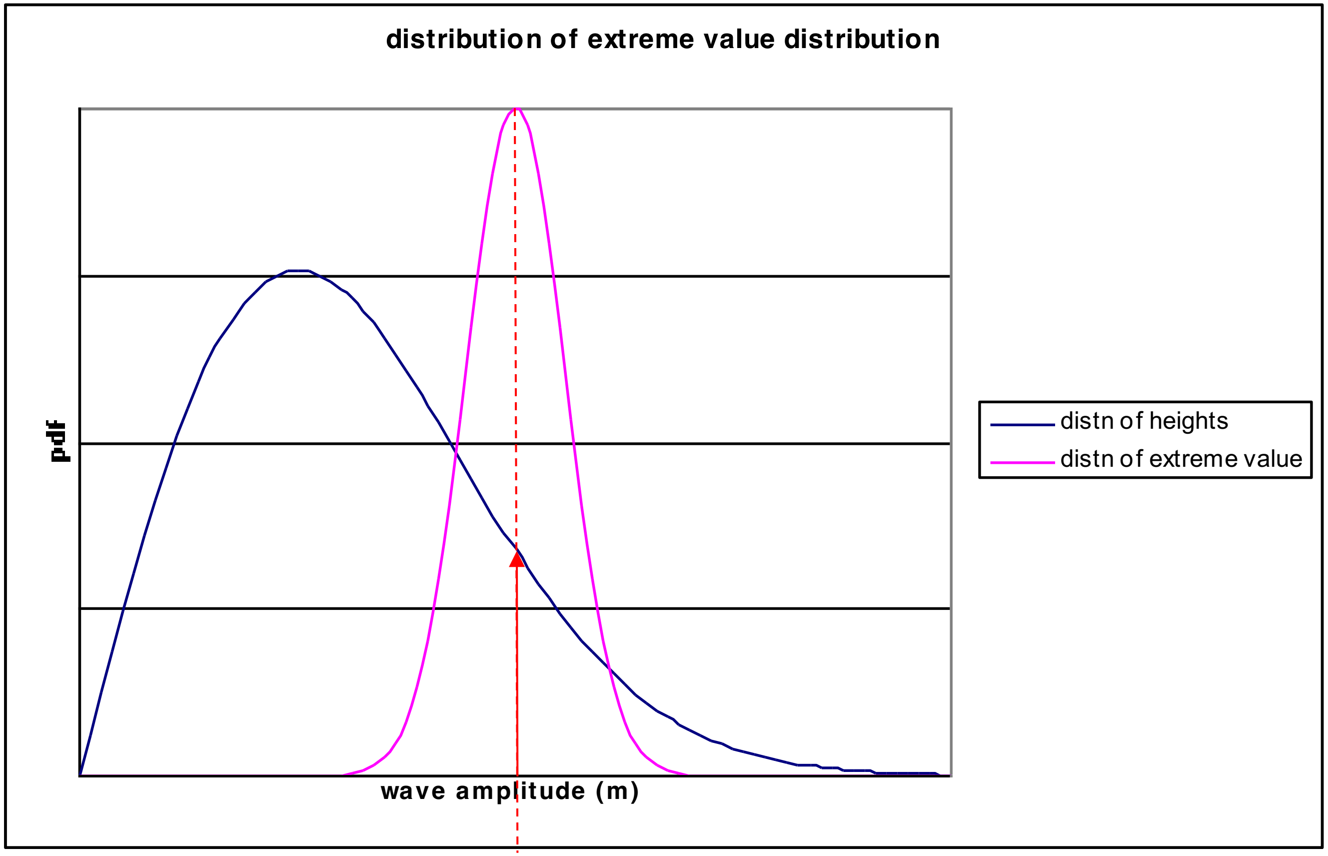
"Generalised gamma" distribution appears to fit most data over range of heights.




see Ochi (1998)

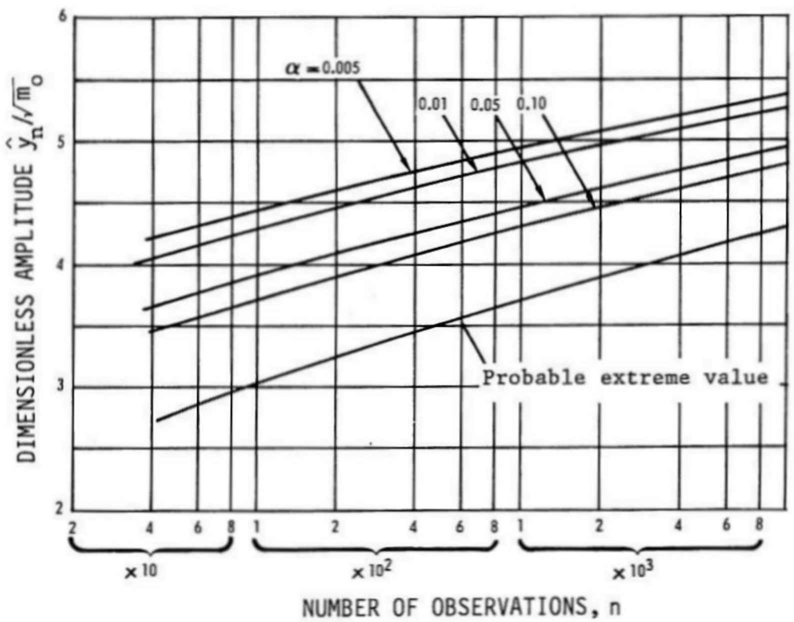
# Estimating extreme wave heights: distribution of a distribution

Important to realise that the predicted extreme height is itself a sample of a population, with its own probability distribution.



# distribution of a distribution

There is a chance that the modal extreme height will be exceeded. Allow for this by introducing a design risk factor alpha (probability of height being exceeded).




Choice of alpha is subjective. A value of 0.01 is often used.

For alpha= 0.01, design height is about 33% > than modal extreme height.

If alpha = 0.005 is selected instead, design wave is only an extra 4% higher than for an alpha of 0.01

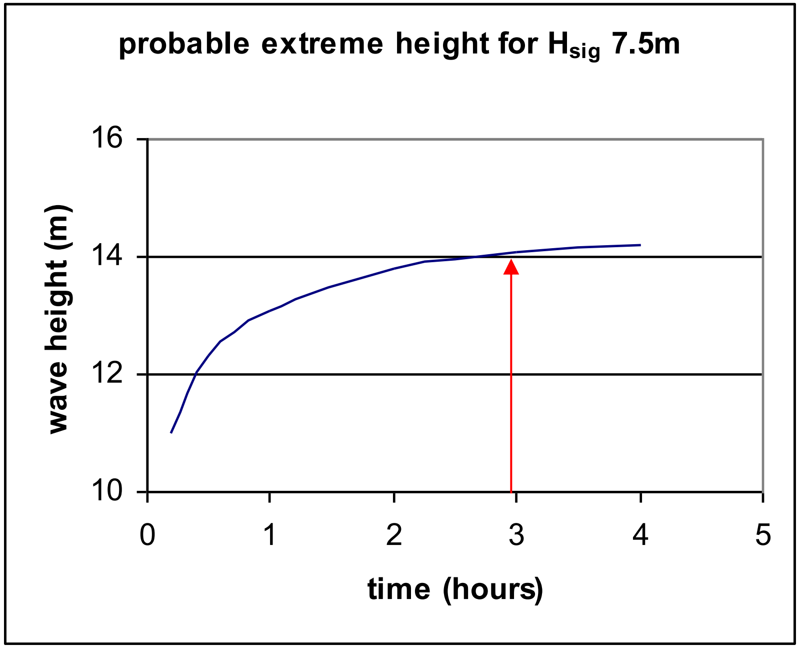
# Estimating extreme wave heights: distribution of a distribution

- All statistical formula so far assume a narrow band spectrum.
- Spectral broadness $\varepsilon$ has a small effect on result.
- However, ocean wave spectra range from $0.45 < \varepsilon < 0.8$ , yielding very small variation of height.
- but be careful if the spectrum is unusual e.g. shallow or confined waters.

# Estimating extreme wave heights: effect of storm duration

The number of events (waves) experienced in an extreme storm also depends on the storm duration.

A storm duration of 3 hours is usually used, but the effect of choosing other values is shown below.



# Summary

- Wave amplitudes have a Rayleigh distribution, not a Gaussian distribution.
- Wave height exceedance probability is found from $m_0$.
- Wave height and period are not independent, so have to use e.g. Longuet Higgins theory of exceedance.
- Exceedance then depends on spectral broadness.
- Long term extreme wave height estimation is inexact: extrapolate best fit to hindcast data.
- Extrapolated wave height itself has a probability of being exceeded – design risk factor.
- Spectral broadness and storm duration influences extreme height.

# 4. Probability distributions Summary

You should now be able to:

- Identify the need to predict short- and long-term wave probability and extreme wave events.
- Explain why probability density functions and error functions are used.
- Show why wave amplitudes have a Rayleigh distribution.
- Calculate exceedance probabilities and joint probabilities of height and period.
- Estimate extreme wave heights.

# Real ocean waves content

1. Irregular waves

   Superposition, significant wave height ✓

2. Wave spectra ✓

   Spectral statistics, stylised spectra, encounter spectra, broadness, directionality

3. Probability distributions and extreme waves ✓

   Rayleigh and Gaussian distributions, extreme wave height probability, joint probabilities

# Waves references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Hogben N. Dacunha, N.M.C. & Oliver, G.F. (1986) Global wave statistics. Compiled by BMT. Unwin. *(One of the most extensive indexes of wave statistics for oceans and times of the year.)*

Ochi, M.K.(1998) Ocean waves. Cambridge Ocean Technology Series no.6, Cambridge University Press.*(This is a thorough treatise on ocean waves.)*